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The Ocean Model

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Version 1.0

A. Introduction

The ocean model and this documentation are based on the version distributed by GFDL (Cox, 1984). The structure of the ocean model itself has been changed to allow it to be coupled to the atmosphere model and to make it more efficient on computers with moderate vector length and without virtual memory.

More than two decades have passed since the first primitive equation, 3-dimensional numerical ocean model was coded for use in studying the most basic aspects of large-scale, baroclinic ocean circulation (Bryan and Cox, 1967). A description of the physics and numerics involved was published by Bryan (1969). In this model the prediction of currents is carried out using the Navier-Stokes equations with three basic assumptions. The Boussinesq approximation is adopted, in which density differences are neglected except in the buoyancy term. The hydrostatic assumption is made in which local acceleration and other terms of equal order are eliminated from the equation of vertical motion. And, lastly, closure is attained by adopting the turbulent viscosity hypothesis in which stresses exerted by scales of motion too small to be resolved by the grid are represented as an enhanced molecular mixing. The temperature and salinity are calculated using conservation equations, again utilizing a turbulent mixing hypothesis for closure. The equations are linked by a simplified equation of state.

Several techniques are used for the purpose of computation efficiency. High speed, external gravity waves are eliminated by the 'rigid-lid' approximation, and a LaPlacian equation is solved for the non-divergent, vertically averaged flow. The next most serious time-step limitation, the half pendulum day constraint associated with inertia-gravity waves, is overcome by a semi-implicit treatment of the Coriolis term.

Considerable improvement was made to the structure of the FORTRAN code of this model by Semtner (1974) who, at the same time, added various features to the mathematical formulation, chief of which was the use of 'hole relaxation' (Takano, 1974) in the solution of the external mode for a model with islands. This version of the model has been adopted by many investigators and has seen considerable use for a number of years in the ocean modelling work at GFDL. During this time, as vector processing machines became more demanding of suitable FORTRAN structure, significant changes have been made to the code for efficiency purposes. It has also been generalised in several ways, among which is the incorporation of variable grid spacing in the horizontal, and an arbitrary number of trace prognostic variables. The relaxation code for the solution of the external mode has been redesigned, and a better technique for establishing the initial guess has reduced the scans-to-convergence considerably.

While many alterations have been made to the original Semtner code, anyone who is familiar with that code will note that the basic structure remains the same. It is the goal to provide improvements while retaining as much continuity with the former version as possible.

The main text of this paper describes the version of the ocean model as distributed by GFDL. A number of enhancements have been made to the model. These are described in separate papers or as new annexes to

B. Continuous Equations of the Model

The basic equations of the model as described above are written here in continuous form. Let

$$\begin{aligned} m &= \sec\phi \\ n &= \sin\phi \\ f &= 2\Omega\sin\phi \\ u &= \frac{a}{m}\frac{d\lambda}{dt} \\ v &= a\frac{d\phi}{dt} \end{aligned} \tag{1}$$

where ϕ is latitude, λ is longitude and a is the radius of the earth. An advection operator,

$$\Gamma(\mu) = ma^{-1}[(u\mu)_\lambda + (v\mu m^{-1})_\phi] + (w\mu)_z \tag{2}$$

is adopted in which μ is any scalar quantity. The equations of motion are then

$$u_t + \Gamma(u) - fv = -ma^{-1}(p/\rho_0)_\lambda + F^u \tag{3}$$

$$v_t + \Gamma(v) + fu = -a^{-1}(p/\rho_0)_\phi + F^v \tag{4}$$

where ρ_0 is taken to be unity. The local pressure, p, is given by the hydrostatic relation,

$$p(z) = p^s + \int_z^0 g\rho\, dz \tag{5}$$

where p^s is the pressure at the surface of the ocean. The continuity equation is

$$\Gamma(1) = 0 \tag{6}$$

The conservation equation,

$$T_t + \Gamma(T) = F^T \qquad 7)$$

applies to any 'tracer' type of quantity carried in the model. These include the active tracers, potential temperature and salinity (active in the sense that they appear in the equation of state), and any passive tracers such as Carbon 14 or Tritium. The equation of state is

$$\rho = \rho(\theta, S, z) \qquad (8)$$

where θ is potential temperature, *S* is salinity and the depth dependence arises from compression effects. In the present model, (8) is represented by a polynomial fit to the Knudsen formula (Bryan and Cox, 1972) or by the newer UNESCO (1981) equation of state, see technical documentation. Let

$$\nabla^2 \mu = m^2 \mu_{\lambda\lambda} + m(\mu_\phi / m)_\phi \qquad (9)$$

Then the effects of turbulent mixing are

$$F^u = A_{MV} u_{zz} + A_{MH} a^{-2}[\nabla^2 u + (1 - m^2 n^2)u - 2nm^2 v_\lambda] \qquad (10)$$

$$F^v = A_{MV} v_{zz} + A_{MH} a^{-2}[\nabla^2 v + (1 - m^2 n^2)v + 2nm^2 u_\lambda] \qquad (11)$$

$$F^T = [(A_{TV}/\delta)T_z]_z + A_{TH} a^{-2} \nabla^2 T \qquad (12)$$

where A_{ab} is the mixing coefficient corresponding to

a	M	momentum
	T	tracer
b	V	vertical
	H	horizontal

Vertical mixing is known to be a rather complex function of vertical stability in nature. Since this process is still not well understood, we have adopted a simple, uniform mixing under statically stable conditions, and an infinite mixing under statically unstable conditions. If ρ^l_z is the local vertical density gradient (ignoring compression effects), then

$$\delta = \begin{matrix} 1 & \rho^l_z < 0 \\ 0 & \rho^l_z > 0 \end{matrix} \qquad (13)$$

(Note: stability dependent mixing is described in a companion paper)

At lateral walls, the boundary conditions are

$$\begin{aligned} u &= 0 \\ v &= 0 \\ T_n &= 0 \end{aligned} \qquad (14)$$

where the *n* subscript indicates a local derivative with respect to the co-ordinate normal to the wall. At the surface,

$$\begin{aligned} \rho_0 A_{MV}(u_z, v_z) &= (\tau^\lambda, \tau^\phi) \\ A_{TV}(T_z) &= \eta \\ w &= 0 . \end{aligned} \qquad \text{at z=0} \qquad (15)$$

The 'rigid-lid' assumption of zero vertical motion at the surface filters out high speed external gravity waves which would otherwise seriously limit the length of the time step of the numerical integration. The quantities τ^λ, τ^ϕ are the zonal and meridional components of the surface stress, and η is a flux through the surface, of the particular tracer involved. At the bottom,

$$\begin{aligned} \rho_0 A_{MV}(u_z, v_z) &= (\tau_B^\lambda, \tau_B^\phi) \\ T_z &= 0 \\ w &= -mua^{-1}H_\lambda - va^{-1}H_\phi . \end{aligned} \qquad \text{at z = -H} \qquad (16)$$

where τ_B^λ and τ_B^ϕ are bottom stresses. (Section E explains how the bottom boundary conditions are

implemented in the model).

Combining (3) and (4) with (5),

$$\begin{aligned} u_t &= u'_t - ma^{-1} p^s_\lambda \\ v_t &= v'_t - a^{-1} p^s_\phi \end{aligned} \tag{17}$$

where

$$u'_t = -\Gamma(u) + fv - mga^{-1} \int_z^0 \rho_\lambda dz' + F^u \tag{18}$$

$$v'_t = -\Gamma(v) - fu - ga^{-1} \int_z^0 \rho_\phi dz' + F^v \tag{19}$$

Let us define

$$\begin{aligned} u &= \hat{u} + \bar{u} \\ v &= \hat{v} + \bar{v} \end{aligned} \tag{20}$$

where

$$\bar{\mu} = H^{-1} \int_H^0 \mu dz \tag{21}$$

Then, since p^s is not a function of depth,

$$\begin{aligned} \hat{u}_t &= u'_t - \overline{u'_t} \\ \hat{v}_t &= v'_t - \overline{v'_t} \end{aligned} \tag{22}$$

Since all terms on the right of (18) and (19) are known, (22) may be solved for the internal modes of momentum. Under the rigid-lid boundary condition, the external mode of momentum may be represented by a volume transport stream function ψ ,

$$\bar{u} = -(aH)^{-1}\psi_\phi$$
$$\bar{v} = m(aH)^{-1}\psi_\lambda \ . \qquad (23)$$

This is shown by integrating (6) vertically, substituting (23) and noting that the boundary conditions (15) and (16) on w are satisfied. A prognostic equation for ψ may be obtained by averaging (17) vertically, and eliminating terms in p^s by applying the curl operator,

$$curl_z(\bar{v}_t, \bar{u}_t) = ma^{-1}(\bar{v}_{t\lambda} - (\bar{u}/m)_{t\phi}) \qquad (24)$$

Substituting (23),

$$[m\psi_{t\lambda}/(Ha^2)]_\lambda + [\psi_{t\phi}/(mHa^2)]_\phi = a^{-1}[\overline{v'}_{t_\lambda} - (\overline{u'_t/m})_\phi] \qquad (25)$$

The boundary condition on ψ at lateral walls, corresponding to (14) is

$$\psi_\phi = \psi_\lambda = 0 \ . \qquad (26)$$

This condition is satisfied by setting ψ constant over each unconnected land mass comprising the ocean boundary. In the case of an enclosed basin with no islands, ψ may be arbitrarily set to zero over the boundary forming land mass. If, in addition, islands are present, the associated constant for each island reflects the net flow around the island and must therefore be predicted by the governing equations. The method used is 'hole relaxation' in which the line integral of the quantity ∇p^s, taken around the island, is required to vanish. Averaging (17) vertically, integrating around the coast of the island and setting the contribution due to p^s to zero, the predictive equation,

$$\oint[\{m\psi_{t\lambda}\}d\phi - \{\psi_{t\phi}/(mH)\}d\lambda] = a\oint[\overline{v'_t}d\phi + (\overline{u'_t/m})d\lambda] \qquad (27)$$

is obtained. Applying the Stokes theorem yields a more useful form,

$$a^{-2}\int_A [\{m\psi_{t\lambda}/H\}_\lambda + \{\psi_{t\phi}/(mH)\}_\phi]\,dA = a^{-1}\int_A [\overline{v'_{t_\lambda}} - (\overline{u'_t/m})_\phi]\,dA \qquad (28)$$

Note that (28) is simply an area integral of (25), taken over the island.

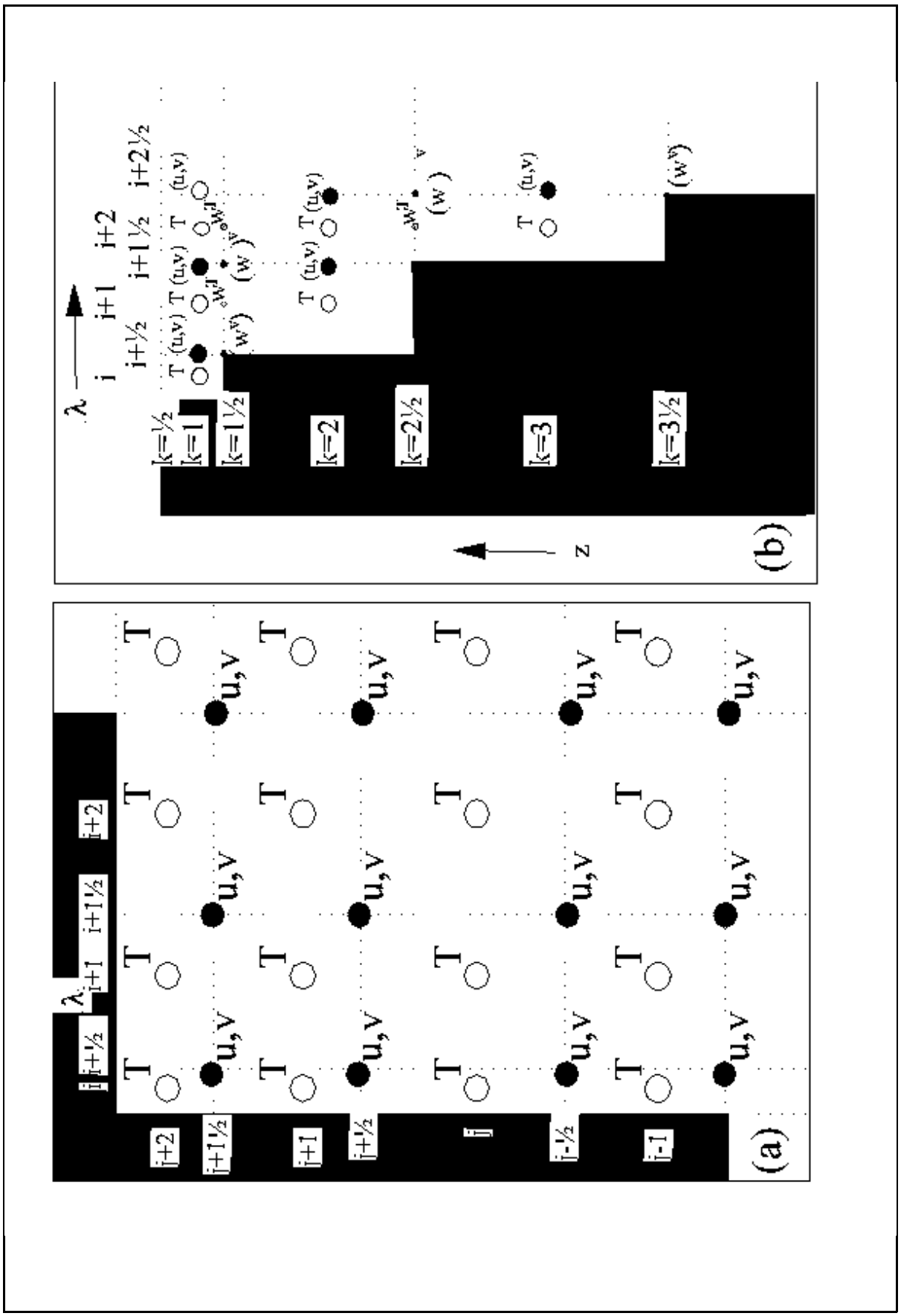

Figure 1 Layout of variables on the model grid.

C. Finite Difference Formulation

The initial value problem described in Section B by prognostic equations (7), (22), (25), associated diagnostic equations (6), (8), (23), and boundary conditions (14), (15), (16), (26), (28) may be solved numerically using finite difference techniques. Rewriting the continuous equations in finite difference form may be done in any of several different ways. However, it is of critical importance that certain integral constraints be maintained during the solution of the initial value problem, and these constraints dictate the particular finite difference formulation which is used. Arakawa (1966) did much of the early work along these lines and Bryan (1969) generalized it using the arguments summarized below.

Let the basin under consideration be divided into cells with interfaces lying along common planes of constant latitude, longitude, and depth. The first constraint which must be satisfied is that of mass conservation within each cell. If each cell interface is designated by the index b, with the area of interface *b* equal to A_b and the velocity normal to interface b into the cell equal to V_b, then

$$\sum_{b=1}^{6} V_b A_b = 0 \ . \tag{29}$$

Secondly, the basin wide integral, I' of any conserved quantity *q*, must remain unchanged by the advective process. This assures conservation of momentum, heat, salt and other tracer quantities. If q_b is the value of *q* on interface *b* and there are a total of *N* cells in the (closed) basin, then

$$\frac{dI'}{dt} = -\sum_{n=1}^{N}\sum_{b=1}^{6} q_b V_b A_b = 0 \ . \tag{30}$$

That the above integral vanishes in general can be seen by considering that each term involving interior interfaces appears twice, once with a positive sign and once with a negative sign. The remaining, uncancelled terms at the boundaries are zero since V_b is zero there.

The third constraint is that the volume integral, *I''* of the square of *q* is unchanged by advection. This assures conservation of kinetic energy and the variance of temperature, salinity and other tracer quantities. If Q_n is the average of *q* within the cell, then

$$\frac{dI''}{dt} = -2\sum_{n=1}^{N}\sum_{b=1}^{6} q_b Q_n V_b A_b = 0. \tag{31}$$

The above integral is not zero for all definitions of q_b , although it has been shown (Arakawa, 1966) that it can be made to vanish by defining

$$q_b = (Q_b + Q_n)/2 \tag{32}$$

where Q_b is the average of *q* in the cell sharing interface *b*. The integral can be rewritten

$$\frac{dI''}{dt} = -\sum_{n=1}^{N}[Q_n^2\sum_{b=1}^{6}V_bA_b + \sum_{b=1}^{6}Q_nQ_bV_bA_b] \tag{33}$$

The first term on the right vanished by (29) and the second term vanishes by the same cancelling process which occurs in (30).

The fourth constraint which must be met by the finite difference equations is that kinetic energy gained (lost) through the pressure term of the momentum equations must be balanced by an equal loss (gain) of potential energy through th advection terms of the conservation equation for density itself, but only for the temperature and salinity separately. Therefore, when the equation of state, (8) is nonlinear, the following balance does not strictly hold. Multiplying the pressure terms by *u* and *v*, the density advection by *gz*, and integrating, we get in the continuous form,

$$\int_V[-ma^{-1}u(p/\rho_0)_\lambda - va^{-1}(p/\rho_0)_\phi]dV = -\int_V gz\Gamma(\rho)dV\ . \tag{34}$$

Finally, since an insulating boundary condition exists for the tracer quantities, *T*, at all boundaries of the basin except the surface, the constraint,

$$\int_V F^T dV = \int_A \eta\, dA \tag{35}$$

must be met, where the integral on the right is taken over the surface area of the basin.

Let the cells described earlier be indexed such that the eastward position is given by *i*, the northward position by *j* and the downward position by *k*. The dimensions of each cell will be denoted by $\Delta_i\lambda$, $\Delta_j\phi$ and $\Delta_k z$. The arrangement of variables within the cells corresponds to the 'B-grid' configuration of Arakawa and Lamb (1977). Horizontally, the tracer quantities, *T* are situated in the centres of the cells with the horizontal

velocity components placed at the corners as shown in Fig. 1a. The two-dimensional quantity, ψ is also positioned at the horizontal centre of the cells. The vertical arrangement is illustrated in Fig. 1b. *T, u* and *v* are located halfway through the vertical dimension of the cell. Two sets of vertical velocities are calculated. The quantity w^T is used for computing *T* and w^v is used for computing *u* and *v*. In each case *w* is calculated at the horizontal interface of the cell, in vertical line with its associated prognostic variable, *T, u, v*.

It will be convenient to define the following finite difference operators:

$$\delta_\lambda(\mu_i) = (\mu_{i+\frac{1}{2}} - \mu_{i-\frac{1}{2}})/\Delta_i\lambda$$

$$\overline{\mu_i}^{\lambda} = (\mu_{i+\frac{1}{2}} + \mu_{i-\frac{1}{2}})/2$$

$$Max_\lambda(\mu_i) = \textit{maximum of } (\mu_{i+\frac{1}{2}}, \mu_{i-\frac{1}{2}})$$

$$Min_\lambda(\mu_i) = \textit{minimum of } (\mu_{i+\frac{1}{2}}, \mu_{i-\frac{1}{2}})$$

In the following discussion, the indices *i,j* will be used in some equations to denote the position of *T* and in other equations to denote the position of *u,v*. Whether *i* and *j* are full integers (*T*) or half integers (*u,v*) will be implied by the variable they index. Further, if indices are omitted, the values *i,j,k* are understood.

Based upon the previous description of the placement of variables within the cells, it is possible to define an alternate cell where the tracer quantities appear at the corners. The horizontal dimensions of such as cell would be

$$\begin{aligned} \Delta_i\lambda &= \overline{\Delta_i\lambda}^{\lambda} \\ \Delta_j\phi &= \overline{\Delta_j\phi}^{\phi} \end{aligned} \tag{36}$$

where, as stated above, the index of the term on the left is implied to be a half integer and that on the right, a full integer. Further, cells can be defined in which *T, u* and *v* appear at vertical interfaces. Their vertical dimension is

$$\Delta_k z = \overline{\Delta_k z}^{z} \tag{37}$$

where, again, the index on the left is implied to be a half integer. These cells are used later in forming volume integrals of quantities appearing at the horizontal or vertical velocity points.

From Fig. 1b it may be seen that the total depth of the basin H^v, defined at the corner points of the original cells is

$$H^{v} = Min_{\lambda}[Min_{\phi}(H^{T})] \tag{38}$$

or the minimum of the depths H^{T}, of the four surrounding vertical columns of cells. We can now define a finite difference equivalent of the vertical averaging operator (21),

$$\bar{\mu}^{*} = H^{-1}\sum_{k=1}^{K}\mu\Delta z \ . \tag{39}$$

Rewriting (20), we get

$$\begin{aligned} u &= \hat{u} + \bar{u}^{*} \\ v &= \hat{v} + \bar{v}^{*} \end{aligned} \tag{40}$$

where (23) is rewritten

$$\begin{aligned} \bar{u}^{*} &= -(aH)^{-1}\delta_{\phi}\overline{\psi}^{\lambda} \\ \bar{v}^{*} &= m(aH)^{-1}\delta_{\lambda}\overline{\psi}^{\phi} \end{aligned} \tag{41}$$

We will first develop the finite difference equations for the internal modes, followed by the equations for ψ and then the equations for the tracers, *T*.

Using centered differencing in time, (22) is written

$$\begin{aligned} \delta_{t}(\hat{u}^{t}) &= \delta_{t}\bar{u}^{,t} - \delta_{t}\overline{\bar{u}^{,t}}^{*} \\ \delta_{t}(\hat{v}^{t}) &= \delta_{t}\bar{v}^{,t} - \delta_{t}\overline{\bar{v}^{,t}}^{*} \end{aligned} \tag{42}$$

The Coriolis term is handled semi-implicitly in time so that the time-step need not be limited to a value small enough to resolve interial oscillations. Equations (18) and (19) may be written

$$\delta_{t}\bar{u}^{,t} - f\alpha v^{,N+1} = G^{u} \tag{43}$$

$$\delta_t \overline{v'}^t + f\alpha v'^{N+1} = G^v \tag{44}$$

where *N+1* indicates the time step being predicted and

$$G^u = -\Gamma^*(u) + f(1-\alpha)v^{N-1} - mga^{-1}\sum_{k'=\frac{1}{2}}^{\overline{k-\frac{1}{2}}^{\lambda}} \delta_\lambda \overline{\rho_{k'}}^z \overline{\Delta_{k'} z}^z + F^{u^*} \tag{45}$$

$$G^v = -\Gamma^*(v) - f(1-\alpha)u^{N-1} - ga^{-1}\sum_{k'=\frac{1}{2}}^{\overline{k-\frac{1}{2}}^{\phi}} \delta_\phi \overline{\rho_{k'}}^z \overline{\Delta_{k'} z}^z + F^{v^*} \tag{46}$$

It may be shown that, for a value of $\alpha > \frac{1}{2}$, the constraint of resolving internal oscillations in time is removed. This is particularly useful in coarse resolution studies where long timesteps are otherwise possible.

Before defining the advective operator Γ^*, we construct two auxiliary quantities

$$u^* = \overline{\hat{u}}^\lambda - (\delta_\phi \psi)/[aMax_\lambda(H)] \tag{47}$$

$$(v/m)^* = \overline{\hat{v}/m}^\phi - (\delta_\lambda \psi)/[aMax_\phi(H)] \tag{48}$$

Then,

$$\Gamma^*(\mu) = ma^{-1}[\delta_\lambda u^* \overline{\mu}^\lambda + \delta_\phi\{(v/m)^* \overline{\mu}^\phi\}] + \delta_z(w^v \overline{\mu}^z) \tag{49}$$

and w^v is defined by

$$\Gamma^*(1) = 0 \tag{50}$$

along with the boundary condition

$$w^{v}_{k=\frac{1}{2}} = 0 \tag{51}$$

Constraints (29) and (30) are satisfied automatically by (50) and constraint (31) is satisfied by expressing the advected quantity, μ at the various interfaces, as the average of the neighbouring values as in (32). The use of the special quantities (47) and (48) is needed so that the vertical velocity at the bottom of the basin is of the form of boundary condition (16).

The frictional terms are lagged one timestep for purposes of numerical stability (Richtmyer and Morton, 1967),

$$F^{u*} = \delta_z(A_{MV}\delta_z u^{N-1}) + A_{MH}a^{-2}[\nabla^{2*}u^{N-1} + (1-m^2n^2)u^{N-1} - 2nm^2\delta_\lambda \overline{v^{N-1}}^{\lambda}] \tag{52}$$

$$F^{v*} = \delta_z(A_{MV}\delta_z v^{N-1}) + A_{MH}a^{-2}[\nabla^{2*}v^{N-1} + (1-m^2n^2)v^{N-1} - 2nm^2\delta_\lambda \overline{u^{N-1}}^{\lambda}] \tag{53}$$

where

$$\nabla^{2*}(\mu) = m^2\delta_\lambda\delta_\lambda\mu + m\delta_\phi(m^{-1}\delta_\phi\mu) \tag{54}$$

The finite difference form of (25) for the stream function, ψ is also complicated somewhat by the semi-implicit treatment of the Coriolis term. Bringing the implicit component to the left side.

$$\begin{aligned}&\delta_\lambda[(m/(\overline{H}^{\phi}a^2)]\delta_\lambda\delta_t\overline{\psi}^{t}] + \delta_\phi[\{1/(m\overline{H}^{\lambda}a^2)\}\delta_\phi\delta_t\overline{\psi}^{t}] \\ -\delta_\phi[\{\alpha f/(Ha)\}\delta_\lambda\overline{\psi^{N+1}}^{\phi}]\Delta\lambda^{\lambda}/\Delta\lambda &- \delta_\lambda[\{\alpha f/(Ha)\}\delta_\phi\overline{\psi^{N+1}}^{\lambda}]\Delta\phi^{\phi}/\Delta\phi \\ &= a^{-1}[\delta_\lambda\overline{G^{v*}}\Delta\phi^{\phi}/\Delta\phi - \delta_\phi(\overline{G^{u*}}/m)\Delta\lambda^{\lambda}/\Delta\lambda]\end{aligned} \tag{55}$$

Boundary condition (26) is satisfied by setting ψ constant along the two rows of cells straddling the basin boundary. In the case of singly connected basin, this value may be arbitrarily set to zero. For an island, the value must be calculated using a finite difference form of (28). This simply amounts to taking an area weighted sum of (55) over all cells for which ψ will take the island value.

The solution of the elliptic problem above is achieved by the method of successive over-relaxation. A guess is

made for all cells variant in ψ, including the island constants, a residual is computed based on (55) and the stated boundary conditions, and a new guess is established from the residual. This process is iterated until the change in ψ between guesses is smaller than an arbitrary constant. Convergence is assured as long as the matrix of coefficients of ψ is diagonally dominant. Large gradients of bottom depth may destroy such dominance through the implicit Coriolis stem of (55). However, this is not a problem for most bottom configurations used.

The predictive equation for the tracer quantities *T* may be written in finite difference form,

$$\delta_t \overline{T}^t = -\Gamma^{**}(T) + F^{T*} \tag{56}$$

where

$$\Gamma^{**}(T) = ma^{-1}[\delta_\lambda\{\overline{u\Delta\phi}^{\phi}\overline{T}^{\lambda}\}/\Delta\phi + \delta_\phi\{\overline{(v/m)\Delta\lambda}^{\lambda}\overline{T}^{\phi}\}/\Delta\lambda] + \delta_z(w^T\overline{T}^z) \tag{57}$$

and w^T is defined by

$$\Gamma^{**}(1) = 0 \tag{58}$$

along with the boundary condition

$$w^T_{k=\frac{1}{2}} = 0$$

Constraints (29), (30) and (31) are satisfied by the same arguments used for the advective operator on momentum. The additional weighting by $\Delta\phi$ and $\Delta\lambda$ under the bar operators is needed to satisfy constraint (34) (see section D).

The diffusive operator may be written

$$F^{T*} = \delta_z(A_{TV}\delta_z T) + (A_{TH}/a^2)\nabla^{2*}T\ . \tag{60}$$

Convective mixing, indicated by $\delta=0$ in (12), is accomplished in the model by testing the vertical static stability of each column of cells at the end of each timestep, volume averaging the *T* values for all cells which are found to be statically unstable, and resetting each cell to the average. This process simulates a vertical

mixing with infinite mixing coefficient.

Two optional forms of time differencing are provided in the model. One solves the vertical diffusion equation using in implicit technique and is described in a separate document. The other solves the joint advection - diffusion equation in the vertical mode and an implicit technique; this is described in Section H.

D. Integral constraints

The importance of maintaining certain constraints on the volume integrals of kinetic and potential energy has been mentioned earlier. It is the purpose of this section to show formally that such constraints are satisfied by the particular set of finite difference equations presented here. The approach follows closely that of Bryan (1969).

If we denote the total kinetic energy (K.E.) by *K*, then

$$K = \bar{K} + \hat{K} \tag{61}$$

where

$$\bar{K} = (\bar{u}^2 + \bar{v}^2)/2 \tag{62}$$

is the K.E. of the external mode and

$$\hat{K} = (\hat{u}^2 + \hat{v}^2)/2 \tag{63}$$

is the K.E. of the internal modes. Let { } designate the volume integral over a closed ocean basin. The left side of (25), when multiplied by $m\psi/H$, integrated over the entire volume and rearranged becomes

$$\{(\frac{m}{a^2 H})[(m\psi\psi_{t\lambda}/H)_\lambda + (\psi\psi_{t\phi}/(mH))_\phi]\} - \{m^2(\psi_\lambda^2)_t/(a^2H^2) + (\psi_\phi^2)_t/(a^2H^2)\}/2$$

The first two terms vanish due to boundary condition (26) on ψ , and the last two are the time derivative of the right side of (62) with (23) substituted. Therefore,

$$\{\bar{K}_t\} = -\{(m\psi/(aH))[\overline{v'_{t\lambda}} - (\overline{u'_t}/m)_\phi]\} \tag{64}$$

by (25) with *u'* and *v'* given by (18) and (19). Furthermore, the individual rates of change by K.E. due to a particular term on the right of (18) and (19) may be obtained by substituting that terms for *u'* and *v'* in (64).

Multiplying (22) by $\hat{u}, \hat{v}$ and integrating,

$$\{\hat{K}_t\} = \{\hat{u}u'_t + \hat{v}v'_t\} - \{\hat{u}\overline{u}_t + \hat{v}\overline{v'}_t\} \ . \tag{65}$$

The second term on the right vanishes by definition of $\hat{u},\hat{v}$. Again, the contributions of the individual terms in (18) and (19) are obtained by substituting them in (65) for *u'*, *v'*.

To obtain an expression for the rate of change of total K.E., it will be necessary to combine the finite difference equivalents of (64) and (65). For this purpose it will be helpful to establish several identities involving the finite difference operators. It is easy to verify that

$$\eta\,\Delta\lambda\,\delta_\lambda\mu + \overline{\mu\Delta\lambda\,\delta_\lambda\eta}^{\lambda} = \Delta\lambda\,\delta_\lambda(\overline{\eta}^{\lambda}\mu) \tag{66}$$

and

$$\overline{\overline{\eta}^{\lambda}\mu}^{\lambda} - \eta\overline{\mu}^{\lambda} = \Delta\lambda\,\delta_\lambda(\mu\Delta\lambda\,\delta_\lambda\eta)/4 \ . \tag{67}$$

Combining (66) and (67) and substituting $\overline{\gamma}^{\phi}$ for μ ,

$$\eta\,\Delta\lambda\,\delta_\lambda\overline{\gamma}^{\phi} + \overline{\gamma\,\overline{\Delta\lambda\,\delta_\lambda\eta}^{\phi}}^{\lambda\phi} = \Delta\lambda\,\delta_\lambda(\overline{\eta}^{\lambda}\overline{\gamma}^{\phi}) + \Delta\phi\,\delta_\phi[\overline{\gamma\,\Delta\phi\,\delta_\phi(\Delta\lambda\,\delta_\lambda\eta)}^{\lambda}]/4 \ . \tag{68}$$

The finite difference equivalent of the right side of (64), corresponding to (55) is

$$\overline{I} = -\sum_i\sum_j\sum_k (ma\psi/H)\,[\delta_\lambda(\overline{\overline{G^{v^*}}^{\phi}}\Delta\phi/\Delta\phi - \delta_\phi(\overline{\overline{G^{u^*}}/m}^{\lambda})\Delta\lambda/\Delta\lambda\,]\Delta z\Delta\phi\Delta\lambda/m \ . \tag{69}$$

Since the integrand is constant in *k*, the vertical summation cancels with *H*. Using (68),

$$\begin{aligned}\overline{I} = a^2\sum_i\sum_j\,&[\overline{(\overline{G^{v^*}}m\,\delta_\lambda\overline{\psi}^{\phi} - \overline{G^{u^*}}\,\delta_\phi\overline{\psi}^{\lambda})/m}^{\lambda\phi}\\ &- \Delta\lambda\,\delta_\lambda(\overline{\psi}^{\lambda}\overline{\Delta\phi\,\overline{G^{v^*}}}^{\phi}) + \Delta\phi\,\delta_\phi(\overline{\psi}^{\phi}\overline{\Delta\lambda\,\overline{G^{u^*}}/m}^{\lambda})\\ &- \Delta\phi\,\delta_\phi[\overline{\overline{G^{v^*}}\Delta\phi\Delta\phi\,\delta_\phi(\Delta\lambda\,\delta_\lambda\psi)}^{\lambda}]/4\\ &+ \Delta\lambda\,\delta_\lambda[\overline{(\overline{G^{u^*}}/m)\Delta\lambda\Delta\lambda\,\delta_\lambda(\Delta\phi\,\delta_\phi\psi)}^{\phi}]/4 \ .\end{aligned} \tag{70}$$

The last four terms on the right vanish by boundary condition (26) on ψ . Substituting (41),

$$\bar{I} = a^2 \sum_i \sum_j \sum_k \overline{H(\bar{u}^* \overline{G^{u*}} + \bar{v}^* \overline{G^{v*}}) \Delta\phi \Delta\lambda / m}^{\lambda\phi} . \tag{71}$$

The finite difference equivalent of the right side of (65), corresponding to (43) and (44) is

$$\hat{I} = a^2 \sum_i \sum_j \sum_k (\hat{u} G^u + \hat{v} G^v) \Delta z \Delta\phi \Delta\lambda / m . \tag{72}$$

Note that, since the integrand is defined at *u*,*v* points, this integral is taken on the alternate cells, with horizontal dimensions described by (36). However, it is easy to show that, for a closed basin, an equivalent expression is

$$\hat{I} = a^2 \sum_i \sum_j \sum_k \overline{(\hat{u} G^u + \hat{v} G^v) \Delta z \Delta\phi \Delta\lambda) / m}^{\lambda\phi} . \tag{73}$$

Combining (71) and (73) we get an expression for the rate of change of total K.E.,

$$I = a^2 \sum_i \sum_j \sum_k \overline{(u G^u + v G^v) \Delta z \Delta\phi \Delta\lambda / m}^{\lambda\phi} \tag{74}$$

where total *u* and *v* are defined by (40). Again, the overbar is optional, making no difference in the value of the integral.

As stated earlier, the rates of change of K.E. due to individual terms of (45) and (46) may be evaluated by substituting them into (74) for G^u and G^v. For advection,

$$I_1 = a^2 \sum_i \sum_j \sum_k [- u\Gamma^*(u) - v\Gamma^*(v)] \Delta z \Delta\phi \Delta\lambda / m . \tag{75}$$

This integral may be rewritten

$$I_1 = -\sum_i \sum_j \sum_k [u \sum_{b=1}^{6} V_b A_b u_b + v \sum_{b=1}^{6} V_b A_b v_b] \tag{76}$$

which is equivalent in form to (31). I_1 has been shown to vanish when u_b, v_b are defined as in (32). This condition is met by the advective operator (49). It is therefore established that no change in total K.E. occurs through advection.

Constraint (34) states that total (kinetic plus potential) energy must be conserved through the pressure term of (45) and (46) and the advection term of (56). Let

$$\gamma = \sum_{k'=\frac{1}{2}}^{k-\frac{1}{2}} \overline{\rho_{k'}}^{z}\, \overline{\Delta_{k'} z}^{z} \qquad (77)$$

$$\delta_z \gamma = \overline{\rho}^{z} .$$

then the rate of change of K.E. by the pressure term is

$$I_2 = ga \sum_i \sum_j \sum_k \overline{[-um\delta_\lambda \overline{\gamma}^{\phi} - v\delta_\phi \overline{\gamma}^{\lambda}]\Delta z \Delta\phi \Delta\lambda / m}^{\lambda\phi} . \qquad (78)$$

Applying (68),

$$\begin{aligned} I_2 = ga \sum_i \sum_j \sum_k [& \gamma \Delta\lambda \delta_\lambda (\overline{u\Delta\phi}^{\phi}) - \Delta\lambda \delta_\lambda (\overline{\gamma}^{\lambda} \overline{u\Delta\phi}^{\phi}) \\ & - \Delta\phi \delta_\phi \overline{[u\Delta\phi \Delta\phi \delta_\phi (\Delta\lambda \delta_\lambda \gamma)]}^{\lambda} / 4 \\ & + \gamma \Delta\phi \delta_\phi (\overline{(v/m)\Delta\lambda}^{\lambda}) - \Delta\phi \delta_\phi (\overline{\gamma}^{\phi} \overline{(v/m)\Delta\lambda}^{\lambda}) \\ & - \Delta\lambda \delta_\lambda \overline{[(v/m)\Delta\lambda \Delta\lambda \delta_\lambda (\Delta\phi \delta_\phi \gamma)]}^{\phi} / 4] \Delta z \end{aligned} \qquad (79)$$

For a closed basin, all terms on the right vanish except for the first and fourth. Combining them and substituting (58),

$$I_2 = ga^2 \sum_i \sum_j \sum_k -\gamma \delta_z (w^T) \Delta z \Delta\phi \Delta\lambda / m . \qquad (80)$$

Using (66) and (77)

$$I_2 = ga^2 \sum_i \sum_j \sum_k \overline{[w^T \Delta z \overline{\rho}^{z}]}^{z} - \Delta z \delta_z (\overline{\gamma}^{z} w^T) \Delta\phi \Delta\lambda / m . \qquad (81)$$

The second term vanishes in the summation, and applying (66) once more with $\eta = z$, .

$$I_2 = ga^2 \sum_i \sum_j \sum_k [-z \Delta z \delta_z (w^T \overline{\rho}^{z}) + \Delta z \delta_z (\overline{z}^{z} w^T \overline{\rho}^{z})] \Delta\phi \Delta\lambda / m . \qquad (82)$$

where, once again, the last term vanishes in the summation. The remainder is equivalent to

$$I_2 = -ga^2 \sum_i \sum_j \sum_k z\Gamma^{**}(\rho)\Delta z \Delta\phi \Delta\lambda / m$$

since the two horizontal terms of (57) vanish in the summation. The expression above is the net loss of potential

The explanation hinges on a rather complicated diagram of the lower portion of the ocean model. For simplicity it has been assumed that there is a one grid box step in the zonal direction which extends north and south (fig. 2).

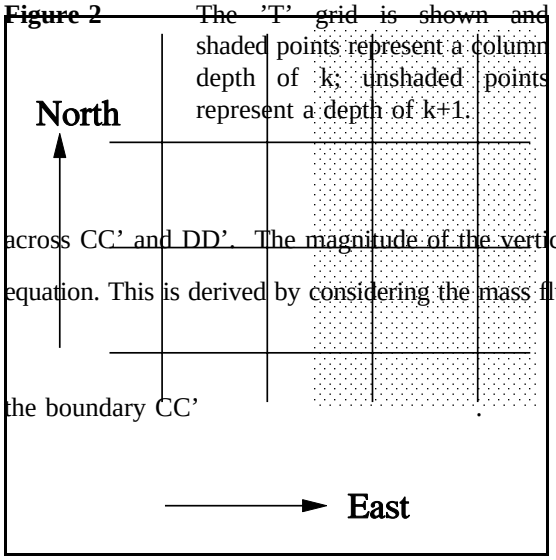

Figure 2 The 'T' grid is shown and shaded points represent a column depth of k; unshaded points represent a depth of k+1.

Looking at the grid 'side on', and looking at fewer points, we see the full, complicated, picture (fig. 3). It is immediately obvious that the total momentum in the box surrounding point $u\nabla h$ is zero, despite momentum flues across CC' and DD'. The magnitude of the vertical velocity across CC' is given implicitly by the continuity equation. This is derived by considering the mass flux across DD' and spreading this across the boundary CC'.

Thus

which recovers the expression . Note that this conserves **mass** but **not** momentum.

The momentum flux across DD' is $(= U_E \Delta Z_{k+1} \rho/2)$, and across CC' is $(w_E = \frac{1}{2} U_E \frac{\Delta z_{k+1}}{\Delta x})$.

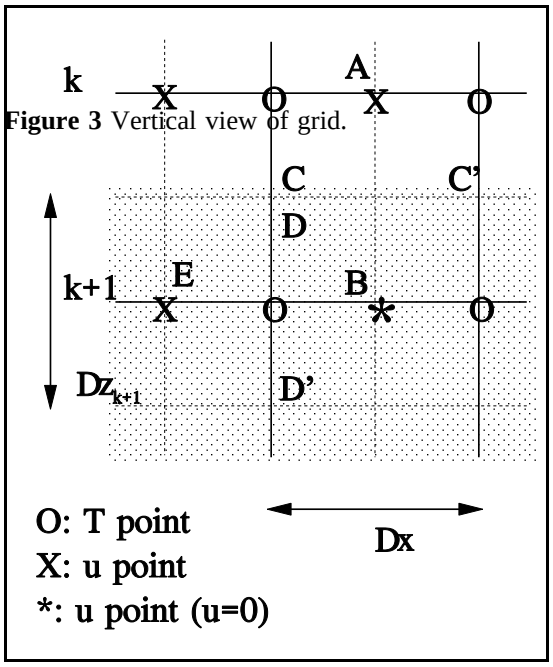

Figure 3 Vertical view of grid.

The ratio of the two is thus $w = \frac{1}{2} U_E \frac{\Delta z_{k+1}}{\Delta x}$.

Momentum is thus **not** conserved: the boundary condition

$$w = u.\nabla h$$

$$U_E^2 \Delta z_{k+1} \rho/4$$

$$U_A w \Delta x \rho/2 = U_E U_A \Delta z_k \rho/4$$

$$\frac{CC'}{DD'} = \frac{U_A}{U_E}$$

represents a source or sink of momentum.

When considering the vertical advection equation, the boundary condition U_B =0 is consistent with this analysis, but does not conserve momentum (needing a boundary layer).

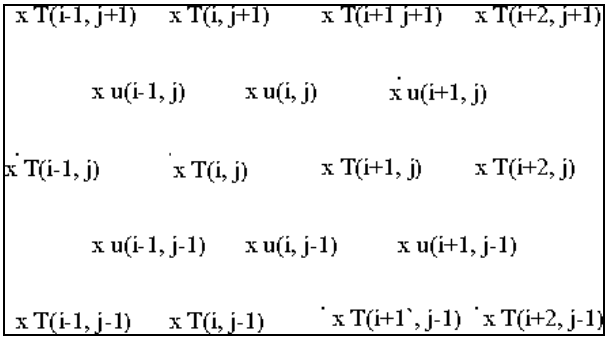
x T(i-1, j+1)
x T(i, j+1)
x T(i+1 j+1)
x T(i+2, j+1)
x u(i-1, j)
x u(i, j)
x u(i+1, j)
x T(i-1, j)
x T(i, j)
x T(i+1, j)
x T(i+2, j)
x u(i-1, j-1)
x u(i, j-1)
x u(i+1, j-1)
x T(i-1, j-1)
x T(i, j-1)
x T(i+1`, j-1)
x T(i+2, j-1)

Figure 4

F. Island boxes

In the standard version of the model released by GFDL in 1984, the stream function field is calculated using an iterative technique. One of the processes involved is to calculate a line integral around each island. (The Unified Model code has generalised the earlier version by allowing each island to consist of several boxes).

The streamfunction is calculated on a T grid.

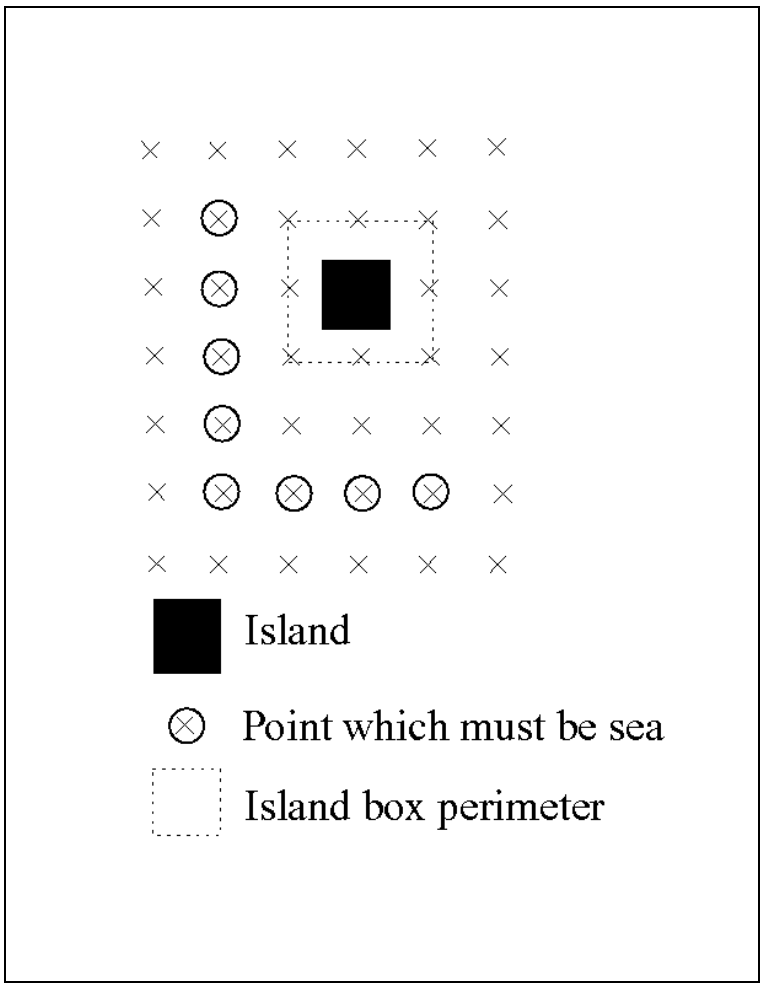

Figure 5 Grid points used in the calculation of stream function.

The number of levels at u(i,j)=min of levels at [(T(i,j),T(i,jk+1),T(i+1,j),T(i+1,u+1)]. An index array is set up to show if a point is land, sea, or coastal:

ISMASK (i,j)
= O if u(i,j), u(i,j-1),u(i-1,j),u(i-1,j-1) are all sea points

= 1 if 1, 2 or 3 of u(i,j), u(i,j-1), u(i-1,j) are all land points

= 2 if u(i,j), u(i,j-1), u(i-1,j), u(i-1,j-1) are all land points

Gridpoints used when calculating the stream function for a point are shown in fig 5. The coefficients CFN, CFW, CFS, CFE are calculated using differences between the grid points at either end of the arrows in fig. 6. PTD is the change in the streamfunction over a timestep.

To calculate the line integral of the streamfunction around an island, residuals are calculated at **all** points with ISMASK=1 (ie at perimeter points of the island) that are within the island box.

Therefore two criteria must be satisfied:

1. The island box must include **all** points with ISMASK=1 that are associated with the island.

2. The island box must **not** include any points with ISMASK=1 that are not associated with the island.

There are two further rules which must be followed in defining island boxes:

1. The island box must be completely outside the island.

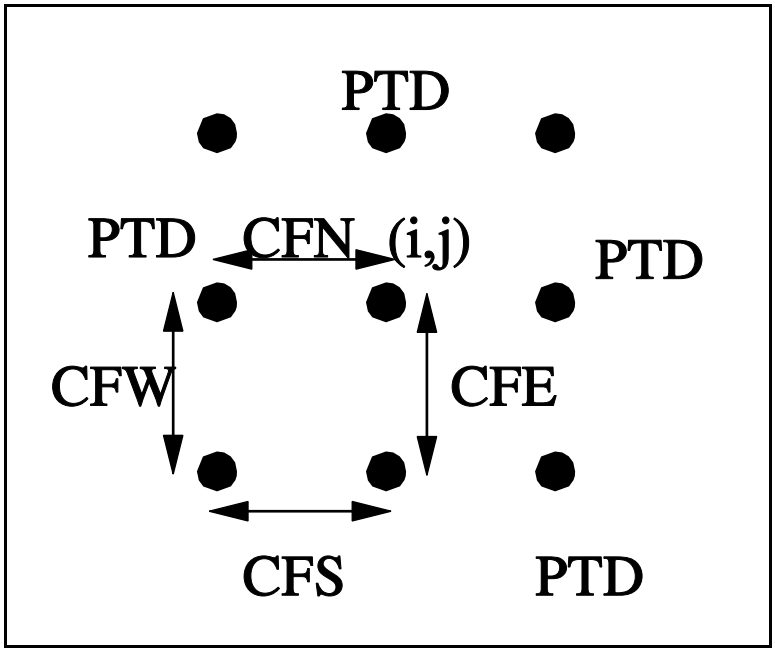

Figure 6 Illustration of terms used in describing island box creation.

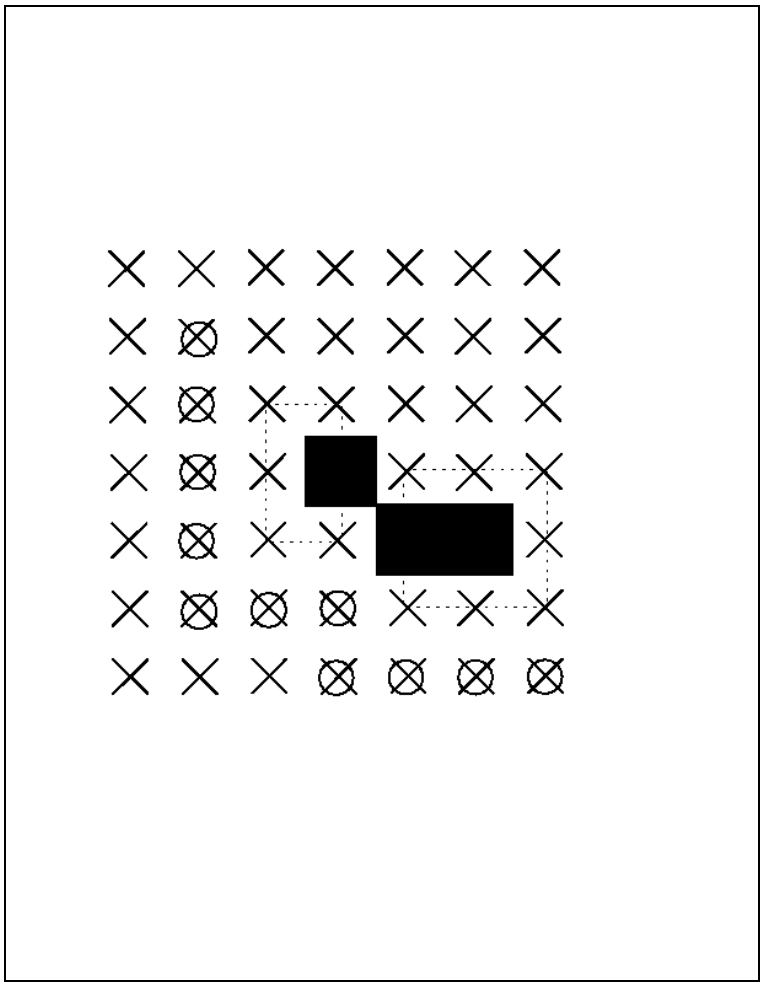

Figure 7 Example choice of island boxes.

2. There must be at least one sea point to the west, to the south and to the southwest of all box boundary points.

This is illustrated in figure 6.

NB (a) If there is only one grid point separating two land masses in the tracer topography, then in the streamfunction topography the two land masses are connected.

(b) It may be necessary to split the island box into more than one segment, each segment satisfying condition (1) above and a slightly modified condition (2).

There must be at least one sea point or another island box segment from the **same** island to the set to the south and to the southwest of all the island segment boundary points.

Also two island box segments from the same island must not overlap. Figure 7 illustrates a valid choice of island boxes.

G. Polynomial approximation to the Knudsen formula

A technique is described in Bryan and Cox (1972) whereby, for each level of the model individually, a 9-term, 3rd order polynomial in temperature and salinity may be constructed to serve as the equation of state, which closely approximates the Knudsen formula for the density of seawater. A considerable decrease in computational effort is realized over using the formula directly. The bounds of T and LS over which the polynomial is to be fitted must be specified for each 250 meter depth span of the ocean. It is currently set at a reasonably general distribution but may be altered if unusual T or S values are expected, such as in a paleo-oceanographic

H. Miscellaneous changes from the original GFDL code

H.1 Robert time filter

The leapfrog scheme can produce ’time splitting’, where solutions for even and odd timesteps drift apart. The Robert time fitter is used to smooth the fields in time using the formula

$$T^{n}_{final} = \nu\,(T^{n+1}_{provisional} + T^{n-1}_{final}) + (1-2\nu)\,T^{n}_{provisional}$$

where the subscript "provisional" refers to the value produced by the GFDL standard code at the timestep given by the superscript, and ’final’ refers to the time-smoothed value.

H.2 Single time-level dumps

To allow a single time level to be stored in the dumps, the ocean model takes a forward timestep occasionally. This should be done frequently enough that a dump output step is associated with a forward timestep. The

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