UNIFIED MODEL DOCUMENTATION PAPER NO. 10

CONSERVATIVE

FINITE DIFFERENCE SCHEMES

FOR A

UNIFIED FORECAST / CLIMATE MODEL

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Version no. 27 dated 29/03/93

This version refers to version 3.1 of the Unified model

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Modification record		
Document version	Author	Description ....

1. INTRODUCTION

This note describes the equations and finite difference formulation of the unified forecast/climate model. The primitive equations are used in the form of White and Bromley (1988), which include certain terms important for treating very large scale flows. The finite difference scheme seeks to combine the accuracy and efficiency of the scheme of Bell and Dickinson (1987) with the conservation properties required for long-term climate integrations. It is a split-explicit scheme on the Arakawa 'B' grid. The Heun time-step is used for advection. Either a second or a fourth order accurate version of the scheme in space can be used. Hybrid vertical coordinates are used, as formulated by Simmons and Strűfing (1981). Mass-weighted linear quantities are conserved, and the mass-weighted second moments of advected quantities are conserved under non-divergent advection using the second order accurate version of the scheme.

To allow use of the model in a pressure coordinate version for middle atmosphere studies, options are included to omit some terms from the equations.

2. THE FORECAST EQUATIONS

The equations set out and justified by White and Bromley (1988) are used. The choice of extra terms to be included is governed by the need for an angular momentum principle and for energy conservation. The notation is that defined in Unified Model Documentation Paper no.5. All quantities are assumed to be in SI units. Define a vertical coordinate $\eta = \eta(p, p_*)$, where $\eta(0, p_*) = 0$ and $\eta(p_*, p_*) = 1$, as in Simmons and Strűfing (1981). The lower boundary of the model is always treated as a material surface with $p=p_*$. Use spherical polar coordinates (λ, ϕ), which for global models will be based on the Earth's axis of rotation. In order that the same model can be used for limited areas, the equations are written for a general position of the coordinate pole. (λ_A, ϕ_A) are used for the actual longitude and latitude, and (λ_0, ϕ_0) are the actual longitude and latitude of the coordinate pole. λ_A at the pole is 0.

The conversion between wind components (u_A, v_A) in actual latitude and longitude and those, (u, v), relative to the coordinate pole is given by

$$u = \cos(ROT)\, u_A + \sin(ROT)\, v_A$$
$$v = \cos(ROT)\, v_A - \sin(ROT)\, u_A \qquad (1)$$

or

$$u_A = \cos(ROT)\, u - \sin(ROT)\, v$$
$$v_A = \cos(ROT)\, u + \sin(ROT)\, v \qquad (2)$$

where

$$\cos(ROT) = \sin\lambda \sin(\lambda_A - \lambda_0)\sin\phi_0 + \cos\lambda\cos(\lambda_A - \lambda_0)$$
$$\sin(ROT) = -\cos\phi_0 \sin(\lambda_A - \lambda_0)/\cos\phi$$

(3)

These relations only appear in the equations through the Coriolis terms.

Write

$$f_1 = 2\Omega\cos\phi_A \sin(ROT)$$
$$= -2\Omega\sin\lambda\cos\phi_0$$
$$f_2 = 2\Omega\cos\phi_A\cos(ROT)$$
$$= 2\Omega\{\cos\phi\sin\phi_0 - \sin\phi\cos\lambda\cos\phi_0\}$$
$$f_3 = 2\Omega\{\sin\phi\sin\phi_0 + \cos\phi\cos\lambda\cos\phi_0\}$$
$$= 2\Omega\sin\phi_A$$

(4)

The equations are then

$$\vartheta u/\vartheta t + (u/r_s\cos\phi)\vartheta u/\vartheta\lambda + (v/r_s)\vartheta u/\vartheta\phi + \eta\vartheta u/\vartheta\eta +$$
$$(l/r_s\cos\phi)\left\{\vartheta\Phi/\vartheta\lambda + \{R(T_v+\mu T_s)/p\}\vartheta p/\vartheta\lambda\right\}$$
$$-f_3 v + f_2\tilde{w} - (u/r_s)(v\tan\phi - \tilde{w}) = F_u$$

(5)

$$\partial v/\partial t + (u/r_s\cos\phi)\partial v/\partial\lambda + (v/r_s)\partial v/\partial\phi + \dot{\eta}\partial v/\partial\eta +$$

$$(1/r_s)\left\{\partial\Phi/\partial\phi + \{R(T_v+\mu T_s)/p\}\partial p/\partial\phi\right\}$$

$$-f_3 u + f_1\tilde{w} - (u^2/r_s)(\tan\phi + v\tilde{w}/r_s) = F_v$$

(6)

$$\partial\theta_L/\partial t + (u/r_s\cos\phi)\partial\phi_L/\partial\lambda + (v/r_s)\partial\phi_L/\partial\phi + \dot{\eta}\partial\phi_L/\partial\eta$$

$$- (1/\pi)\left\{\left\{L_c q_c^{(L)} \; (L_c+L_f)q_c^{(F)}\right\}/(c_{pT})\right\}(RT\omega/c_p p) = F_\phi$$

(7)

$$\partial q_T/\partial t + (u/r_s\cos\phi)\partial q_T/\partial\lambda + (v/r_s)\partial q_T/\partial\phi + \dot{\eta}\partial q_T/\partial\eta = F_q$$

(8)

$$\partial/\partial\eta(r_s^2\partial p/\partial t) + (1/\cos\phi)\{\partial/\partial\lambda(ur_s\partial p/\partial\eta) + \partial/\partial\phi(vr_s\cos\phi\partial p/\partial\eta)\}$$

$$+ \partial/\partial\eta(\dot{\eta}r_s^2\partial p/\partial\eta) = 0$$

(9)

The quantities F_u, F_v, F_θ, F_q are source terms, and also include any diffusion required for computational reasons. The thermodynamic variables θ_L and q_T are given by:

$$\theta_L = \theta - \left(L_c q_c^{(L)} + (L_c+L_f)q_c^{(F)} \right) / (c_p\Pi)$$

(10)

$$q_T = q + q_c^{(L)} + q_c^{(F)} \tag{11}$$

The vertical boundary conditions are:

$$\dot{\eta} = 0 \text{ at } \eta = 0,1 \qquad (12)$$

These boundary conditions are used whether or not the model is integrated over the full depth of the atmosphere.

Integrating (9) in the vertical from $\eta = 0$ to 1 gives:

$$r_s^2(p_*)\,\partial p_*/\partial t = -(1/\cos\phi)\int_0^1 \Big[\,\partial/\partial\lambda\,(u r_s\,\partial p/\partial\eta) + \partial/\partial\phi\,(v r_s \cos\phi\,\partial p/\partial\eta)\Big]\,d\eta$$

(13)

Integrating (9) from $\eta = 0$ to η gives:

$$r_s^2\,\dot{\eta}\,\partial p/\partial\eta = -r_s^2\,\partial p/\partial t - (1/\cos\phi)\int_0^1\Big[\partial/\partial\lambda\,(u r_s\,\partial p/\partial) + \partial/\partial\phi\,(v r_s \cos\phi\,\partial p/\partial\eta)\Big]\,d\eta$$

(14)

The hydrostatic relation is given by:

$$\partial\Phi/\partial\eta = -\{R(T_v + \mu T_s)/p\}\,\partial p/\partial\eta$$
$$= -c_p(\theta_s)\,\partial\Pi/\partial\eta$$

(15)

where $\Pi = (p/100000)^{\kappa}$ and θ_v is the virtual potential temperature which is defined by the standard formula $\theta(1 + (\epsilon^{-1} - 1)q)$, where ϵ is the ratio of molecular weights of water and dry air. The virtual temperature T_v is similarly defined. The basic state temperature T_s and potential temperature θ_s are functions of pressure only, taken from the standard atmosphere defined in Appendix 2. Equation (15) is integrated from $\eta = 1$ to η, with boundary condition $\Phi = \Phi_*$ at $\eta = 1$. Φ is either the specified topographic height, or the specified height of a material surface within the atmosphere which will be acting as a lower boundary for the model. In the latter case, the specified height will normally be derived from that of a constant pressure surface.

The approximate vertical velocity $\tilde{w}$ is defined as

$$\tilde{w} = -(RT_s/gp)\left\{-(1/r_s^2\cos\phi)\int_0^{\eta}\left[\partial/\partial\lambda\,(ur_s\partial p/\partial\eta) + \partial/\partial\phi\,(vr_s\cos\phi/\partial p/\partial\eta)\right]d\eta + (u/r_s\cos\phi)\,\partial p/\partial\lambda + (v/r_s)\,\partial p/\partial\phi\right\}$$

(16)

The pseudo-radius r_s is defined as

$$r_s(p) = a + \int_p^{p_0} RT_s(p')/(gp')\,dp'$$

(17)

where a is the mean radius of the earth at the reference pressure $p_0=101325$ pa

The quantity æ is defined by

$$\mu = \{f_2u - f_1u + (u^2+v^2)/r_{s0}\} / g$$

(18)

3. THE INTEGRATION SCHEME

The variables are held on the Arakawa 'B' grid as in Bell and Dickinson (1987). The variables $u,\ v,\ \theta,\ q$ and è are held at levels Φ , where η_k is the vertical grid-length index, while $\dot{\eta}$ is held at the intermediate levels $\eta_{k+1/2}$ The lower boundary is at $k=\frac{1}{2}$ and the upper boundary at $k=TOP+\frac{1}{2}$. The pressure is defined at intermediate levels by

$$p_{k+1/2} = A_{k+1/2} + B_{k+1/2} p_*$$

(19)

where $A_{k+1/2}$ and $B_{k+1/2}$ are specified constants. Swinbank (1989) has proposed a method of choosing these constants for a 20 level model. Thus

$$(\partial p / \partial p_*)_{k+1/2} = B_{k+1/2} \qquad (20)$$

and

$$\Delta p_k = (A_{k+1/2} - A_{k-1/2}) + (B_{k+1/2} - B_{k-1/2}) p_*$$

(21)

Note that this definition makes Δp_K negative, since k increases with physical height. The pressure at full levels, where required, is defined as

$$p_k = A_k + B_k p_* \qquad (22)$$

where A_k, B_k are chosen so that $\Pi_k = \dfrac{(\Pi_{k+1/2} p_{k+1/2} - \Pi_{k+1/2} p_{k-1/2})}{(\kappa+1)\Delta p_K}$

when $p_*=100000$. This gives a value approximately midway in pressure. Note that in the middle atmosphere version of the model all B_k are zero. A split explicit integration scheme is used, similar to that in Bell and Dickinson (1987). The solution procedure is split into two parts, called the 'adjustment' and 'advection' steps. The adjustment timestep is written as δt , the advection timestep as Δt . In the former, the pressure, temperature, and wind fields are updated using the pressure gradient and the main part of the Coriolis terms, and the vertical advection of potential temperature. Only the final updated values of surface pressure and horizontal wind are used in the next step. The average horizontal wind from the adjustment step is used to define the horizontal advection in the advection step, and, via the continuity equation, the vertical advection. This procedure is needed to ensure conservation. All advection increments are then calculated in the advection step, together with the horizontal diffusion, divergence damping, and the remainder of the Coriolis terms. In the standard configuration of the unified model, it is common for the winds to be much stronger at the top level than the remaining levels. An option is therefore included to halve the timestep at the top level. It is not possible to enforce conservation with respect to time integration when this is done.

The standard finite difference notation

$$\delta_\lambda X = (X(\lambda+\frac{1}{2}\Delta\lambda) - X(\lambda-\frac{1}{2}\Delta\lambda)) / \Delta\lambda$$

$$\overline{X}^\lambda = \frac{1}{2}(X(\lambda+\frac{1}{2}\Delta\lambda) + X(\lambda-\frac{1}{2}\Delta\lambda))$$

is used.

3.1 The adjustment step

This uses a 'forward-backward' scheme in which a forward step is used for the u and v equations, and the new values of these variables are then used in the p_* and θ equations. The 'forward' part of the integration scheme is:

$$u_k^{n+1} = u_k^n + \delta t \left[\frac{1}{2}F^n(v_k^n + v_k^{n+1}) - \overline{\frac{1}{r_s\cos\phi}\left\{\delta_\lambda\Phi_k + \frac{C_p\overline{(\theta+\mu\theta_s)_k}^\lambda}{(K+1)}\delta_\lambda\frac{\left[\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}\right]^n}{\Delta p_K}\right\}}^\phi\right]$$

(23)

$$v_k^{n+1} = v_k^n - \delta t \left[\frac{1}{2}F^n(u_k^n + u_k^{n+1}) + \overline{\frac{1}{r_s}\left\{\delta_\phi\Phi_k + \frac{C_p\overline{(\theta_v+\mu\theta_s)_k}^\phi}{(\kappa+1)}\delta_\phi\frac{\left[\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}\right]^n}{\Delta p_k}\right\}}^\lambda\right]$$

(24)

where

$$F^n = (f + u^n \tan\phi / r_s) \tag{25}$$

and f is the Coriolis parameter $2\Omega\sin\phi_A$. The small Coriolis terms associated with the vertical velocity will be added in the advection step. As in Bell and Dickinson (1987), equations (23) and (24) can be arranged to allow explicit integration. Write

$$\frac{\vartheta\Phi_k}{\vartheta\lambda} = \frac{1}{r_s\cos\phi}\overline{\left\{\delta_\lambda\Phi_k^n + \left(\frac{C-p\overline{(\theta_v+\mu\theta_s)_k}^\lambda}{(\kappa+1)}\right)\delta_\lambda\frac{\left[\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}\right]^n}{\Delta p_k}\right\}}^\phi$$

$$\frac{\vartheta\Phi_k}{\vartheta\phi} = \frac{1}{r_s}\overline{\left\{\delta_\phi\Phi_k^n + \left(\frac{C-p\overline{(\theta_v+\mu\theta_s)_k}^\lambda}{(\kappa+1)}\right)\delta_\phi\frac{\left[\Pi_{k+\frac{1}{2}}p_{k+\frac{1}{2}} - \Pi_{k-\frac{1}{2}}p_{k-\frac{1}{2}}\right]^n}{\Delta p_k}\right\}}^\phi$$

Then eliminating u_k^{n+1} from equations (23) and (24) yields

$$v_k^{n+1} = \frac{\left[v_k^n\left(1 - \frac{\delta t^2}{4}F^{n2}\right) - \frac{\delta t}{2}F^n\left(2u_k^n - \delta t\frac{\vartheta\Phi}{\vartheta\lambda}\right) - \delta t\frac{\vartheta\Phi}{\vartheta\Phi}\right]}{\left(1 + \frac{\delta t^2}{4}F^{n2}\right)}$$

and equation (23) is used to give u_k^{n+1}.

The hydrostatic equation is approximated by

$$\Phi_k = \Phi_* - \sum_{m=1}^{k-1} C_p(\theta_{vm}+\mu\theta_s)(\Pi_{m+1/2}-\Pi_{m-1/2}) + C_p(\theta_{vk}+\mu\theta_s)\left(\Pi_{k-1/2} + \frac{(\Pi_{k-1/2}p_{k-1/2} - \Pi_{k+1/2}p_{k+1/2})}{(\kappa+1)\Delta p_k}\right)$$

(26)

The special form of the last term is chosen to ensure angular momentum conservation.

The ‘backward’ part of the integration scheme is given by

$$p_*^{n+1} = p_*^n + \frac{\delta t}{r_s^2 (p_*)^n} \sum_{m=1}^{TOP} D_m^{n+1} \tag{27}$$

(This update is not performed in the middle atmosphere version).

$$\theta_k^{n+1} = \theta_k^n - \frac{\delta t}{2(r_s^2 \Delta p)_{k^n}} \left[\left(r_s^2 \dot{\eta} \frac{\vartheta p}{\eta \vartheta} \right)_{k+\frac{1}{2}}^{n+1} (\theta_{Rk+1} - \theta_{Rk}) + \left(r_s^2 \dot{\eta} \frac{\vartheta p}{\vartheta \eta} \right)_{k-\frac{1}{2}}^{n+1} (\theta_{Rk} - \theta_{Rk-1}) \right] \tag{28}$$

where $\phi_R(\eta)$ is a basic state profile of ϕ. Note that it is not the same as the basic state ϕ_s used to calculate the pseudo-radius in (15). ϕ_R has to be chosen to be more statically stable than any profile actually present, as when a basic state is extracted for a semi-implicit integration scheme.

$$\left(r_s^2 \dot{\eta} \frac{\vartheta p}{\vartheta \eta} \right)_{k+\frac{1}{2}} = - \left(\frac{r_s^2 (p_{k+1/2})}{r_s^2 (p)} \quad \frac{\vartheta p_{k+1/2}}{\vartheta p_*} \right) \sum_{m=1}^{TOP} D_m + \sum_{m=k+1}^{TOP} D_m$$

(29)

The first term on the right hand side is omitted from the middle atmosphere version.

$$D_m = \frac{1}{\cos\phi} - \left[\overline{\delta \lambda (u_m \overline{r_s \Delta p_m}^{\lambda\phi})}^{\phi} + \delta_\phi \overline{(v_m \overline{r_s \Delta p_m}^{\lambda\phi} \cos\phi)}^{\lambda} \right]$$

(30)

In order to ensure that θ and q are conserved under advection, it is necessary that all advection is done by a three-dimensional velocity field which satisfies the continuity equation. The average fields of $r_s u_m \Delta p_m$ and $r_s v_m \Delta p_m$ over the adjustment steps must be saved for use in the advection step.

At the end of the adjustment steps, the value of θ is rest to its value

at the beginning of the timestep, the update is then recalculated in the advection step.

A smoothing of the averaged wind fields $r_s u_m \Delta p_m$ and $r_s v_m \Delta p_m$ that are used in the advection step may be implemented. On the final adjustment timestep, the velocity fields are smoothed according to

$$\underline{u}^{-n} = \underline{u}^{n} + COEFF^{*}\,({}^{n}C_{r}\underline{u}^{r})$$

where the binomial coefficients ${}^{n}C_{r}$, depend on the number n of adjustment timesteps. This may be interpreted as a time smoothing centered about time $t + \Delta t/2$. The coefficient $COEFF$ is chosen to be 0.05 which has been found to be optimal for time smoothing applied to other schemes.

3.2 Grid splitting

The finite difference scheme described above suffers from grid separation, since the ‘B’ grid supports two independent solutions for gravity waves. To prevent it, a scheme described by Mesinger (1973) and extended by Janjic (1979) can be used. A term of the form

$$w\,\delta t\,\frac{\Delta p_m}{\left(\sum_{i=1}^{TOP}\Delta p_i\right)}\sum_{i=1}^{TOP} r_s^{2}(\nabla_+^2-\nabla_x^2)\,P_i \qquad (31)$$

would be subtracted from D_m (equation (30)) on each adjustment step, where

$$r_s^2\nabla_+^2 P_k = \frac{1}{\cos^2\phi}\left\{\delta_\lambda\left[\overline{\Delta p_k}^{\lambda}\delta_\Phi\lambda_k\right] + \delta_\lambda\left(\delta\, c_p\overline{\theta_k}^{\lambda}\,\overline{\Delta p_k}^{\lambda}\,\delta_\lambda\,[\Delta(\Pi p)_k/\Delta p_k]\right)/(1+\kappa)\right\} + \frac{1}{\cos\phi}\left\{\delta_\phi\left[\overline{\Delta p_k}^{\phi}\cos\phi\,\delta_\phi\Phi_k\right] + \delta\phi\left(c_p\overline{\theta_k}^{\phi}\,\overline{\cos\phi}^{\lambda}\,\overline{\Delta p_k}^{\phi}\,\delta_\phi[\Delta(\Pi p)/\Delta p]_k\right)/(1+\kappa)\right\}$$

(32)

This scheme was found to produce unrealistic pressure builds over the poles, particularly the south pole, during climate integrations. The use of fourth and higher order diffusion was found to remove the grid-splitting mode and so this is used instead.

3.3 The advection step

At the beginning of the advection step, equations (10) and (11) are used to convert θ and q into θ_L and q_T. The Heun advection scheme is used. Experiments within the split-explicit model described by Bell and Dickinson (1987) have shown that it is more stable than the Lax-Wendroff scheme used in that model, even though it has growing eigensolutions of order $(1+O(\Delta t^4))$.

Experiments, Marshall (1989), have shown that the scheme can be corrected to remove this instability. In practice it is found that this correction is submerged in the diffusion required for other reasons, and the correction is therefore not used in the model. The scheme has two steps. The advecting velocity for both is the average value saved from the adjustment steps. Mass-weighted increments to θ_L and q_T have to be predicted to ensure conservation.

Define

$$\underline{U}_k = (U_k, V_k) = (u_k \overline{r_s \Delta p_k}^{\lambda\phi},\ v_k \overline{r_s \Delta p_k}^{\lambda\phi} \cos\phi)$$

(33)

as saved from the adjustment steps.

Define

$$\left(r_s^2 \dot{\eta} \frac{\partial p}{\partial \eta} \right)_{k+\frac{1}{2}} \equiv E_{k+\frac{1}{2}} \tag{34}$$

where $E_{k+\frac{1}{2}}$ is calculated from the finite difference formula (29) using (30) and the grid splitting correction (31). The finite difference equations for the first advection step are then:

$$(r_s^2 \Delta p_k)^n\, \theta_{Lk}^{\#} = (r_s^2 \Delta p_k)^n\, \theta_{Lk}^n -$$

$$\frac{\Delta t}{\cos\phi} \left\{ (1+\nu)\ \overline{\overline{U}_m^{\phi} \delta_\lambda \theta_{Lk}}^{\lambda} - \nu\ \overline{\overline{U}_m^{\phi} \delta_\lambda \theta_{Lk}}^{3\lambda} + (1+\nu)\ \overline{\overline{V}_m^{\lambda} \delta_\phi \theta_{Lk}}^{\phi} - \nu\ \overline{\overline{V}_m^{\lambda} \delta_\phi \theta_{Lk}}^{3\phi} \right\}$$

$$- \frac{1}{2} \Delta t \left\{ E_{k+\frac{1}{2}} (\theta_{Lk+1} - \theta_{Lk}) + E_{k-\frac{1}{2}} (\theta_{Lk} - \theta_{Lk-1}) \right\}$$

(35)

$$- \theta_k (p_*^{n+1} - p_*^n)/4 \ (B_{k+1/2}(3r_{sk} + r_{sk+1})(r_{sk}-r_{sk+1}) - B_{k-1/2}(3r_{sk} + r_{sk-1})(r_{sk}-r_{sk-1}))$$

The term involving θ, p_* , and r_s comes from the vertical velocity term as shown in section 4.2.

The contributions to this term at $k=1/2$ and $k=N+1/2$ are zero as the vertical velocity at these levels are zero. This term is refered to in the code as BRSP since no meaningful name is apparent.

$$(r_s^2 \Delta p_k)^n \, q_{Tk}^{\#} = (r_s^2 \Delta p_k)^n \, q_{Tk}^n -$$

$$\frac{\Delta t}{\cos\phi} \left\{ (1+\nu) \ \overline{\overline{U_m}^{\phi} \delta_\lambda q_{Tk}}^{\lambda} - \nu \ \overline{\overline{U_m}^{\phi} \delta_\lambda q_{Tk}}^{3\lambda} + (1+\nu) \ \overline{\overline{V_m}^{\lambda} \delta_\phi q_{Tk}}^{\phi} - \nu \ \overline{\overline{V_m}^{\lambda} \delta_\phi q_{Tk}}^{3\phi} \right\}$$

$$- \frac{1}{2} \Delta t \left\{ E_{k+\frac{1}{2}} (q_{Tk+1} - q_{Tk}) + E_{k-\frac{1}{2}} (q_{Tk} - q_{Tk-1}) \right\}$$

$$- q_T (p_*^{n+1} - p_*^n)/4 \ (B_{k+1/2}(3r_{sk} + r_{sk+1})(r_{sk}-r_{sk+1}) - B_{k-1/2}(3r_{sk} + r_{sk-1})(r_{sk}-r_{sk-1}))$$

(36)

$$(\overline{r_s^2 \Delta p}^{\lambda\phi})_k^n \, u_k^{\#} = (\overline{r_s^2 \Delta p_k}^{\lambda\phi})^n \, u_k^n -$$

$$\frac{\Delta t}{\cos\phi} \left\{ (1+\nu) \ \overline{\overline{\overline{U_m}^{\lambda\phi} \delta_\lambda}^{\phi} u_k}^{\lambda} - \nu \overline{\overline{\overline{U_m}^{\lambda\phi} \delta_\lambda}^{\phi} u_k}^{3\lambda} + (1+\nu) \overline{\overline{\overline{V_m}^{\lambda\phi} \delta_\phi}^{\lambda} u_k}^{\phi} - \nu \overline{\overline{\overline{V_m}^{\lambda\phi} \delta_\phi}^{\lambda} U_k}^{3\phi} \right\}$$

(37)

$$- \frac{1}{2} \Delta t \left\{ \overline{E_{k+\frac{1}{2}}}^{\lambda\phi} (u_{k+1} - u_k) + \overline{E_{k-\frac{1}{2}}}^{\lambda\phi} (u_k - u_{k-1}) \right\}$$

$$- u_k(\overline{p_*^{n+1}}^{\lambda\phi} - \overline{p_*^{n}}^{\lambda\phi})/4 \;\; (B_{k+1/2}(3\overline{r_{sk}}^{\lambda\phi} + \overline{r_{sk+1}}^{\lambda\phi})(\overline{r_{sk}}^{\lambda\phi} - \overline{r_{sk+1}}^{\lambda\phi}) -$$

$$B_{k-1/2}(3\overline{r_{sk}}^{\lambda\phi} + \overline{r_{sk-1}}^{\lambda\phi})(\overline{r_{sk}}^{\lambda\phi} - \overline{r_{sk-1}}^{\lambda\phi}))$$

$$(\overline{r_s^2\Delta p}^{\lambda\phi})_k^n \, v_k^{\#} = (\overline{r_s^2\Delta p}^{\lambda\phi})_k^n \, v_k^n -$$

$$\frac{\Delta t}{\cos\phi}\left\{(1+\nu)\,\overline{\overline{\overline{U_m}^{\lambda\phi}\delta_\lambda}^{\phi} v_k}^{\lambda} - \nu\overline{\overline{\overline{U_m}^{\lambda\phi}\delta_\lambda}^{\phi} v_k}^{3\lambda} + (1+\nu)\overline{\overline{\overline{V_m}^{\lambda\phi}\delta_\phi}^{\lambda} v_k}^{\phi} - \nu\overline{\overline{\overline{V_m}^{\lambda\phi}\delta_\phi}^{\lambda} v_k}^{3\phi}\right\}$$

(38)

$$-\frac{1}{2}\Delta t\left\{\overline{E_{k+\frac{1}{2}}}^{\lambda\phi}(v_{k+1}-v_k) + \overline{E_{k-\frac{1}{2}}}^{\lambda\phi}(v_k-v_{k-1})\right\}$$

$$- v_k(\overline{p_*^{n+1}}^{\lambda\phi} - \overline{p_*^{n}}^{\lambda\phi})/4 \;\; (B_{k+1/2}(3\overline{r_{sk}}^{\lambda\phi} + \overline{r_{sk+1}}^{\lambda\phi})(\overline{r_{sk}}^{\lambda\phi} - \overline{r_{sk+1}}^{\lambda\phi}) -$$

$$B_{k-1/2}(3\overline{r_{sk}}^{\lambda\phi} + \overline{r_{sk-1}}^{\lambda\phi})(\overline{r_{sk}}^{\lambda\phi} - \overline{r_{sk-1}}^{\lambda\phi}))$$

Note that in the scheme of Bell and Dickinson (1987), higher order accuracy can be achieved by only modifying the second advection step. In the Heun scheme, it is necessary to use the same finite difference approximation in both steps, or else there is an $0(\Delta t^2)$ instability. The value $\nu = 1/6$ in equations (35) to (38) gives fourth order accuracy, but will increase the squared amplification rate of the growing solution from $(1+\frac{1}{4}\xi^4)$ to $(1+\frac{1}{4}\xi_1^4)$ where ξ is the Courant number and $\xi_1 = 1.37\xi$. This will reduce the maximum timestep that can safely be used. A fixed value must be used for ν for each line of latitude in the λ differencing and for each line of longitude in the ϕ differencing to allow conservation. Therefore set

$$\nu = \nu_b(1-\xi_{MAX}^2)$$

where ν_b is an input basic value.

$\xi_{MAX} = MAX\ \xi$ over a line of latitude or longitude as appropriate. If ν_b is given as zero then the code does not calculate ν or the terms in the advection equation associated with ν reducing the execution time by about 30%. The minimum permitted ν value is zero giving second order accurate space advection. ξ is calculated from

$$\xi = \left\{ \frac{u^2 \Delta t^2}{r_s^2 \Delta\lambda^2 \cos^2\phi} + \frac{v^2 \Delta t^2}{r_s^2 \Delta\phi^2} \right\}^{\frac{1}{2}} \tag{39}$$

The second advection step can be written:

$$(r_s^2 \Delta p)_k^{n+1}\ \theta_{Lk}^{n+1} = (r_s^2 \Delta p)_k^{n+1} \theta_{Lk}^{n} - \frac{1}{2}\Delta t \left((\Delta p_k^{n+1} / \Delta p_k^{n})\ \underline{U} \bullet \nabla \theta_{Lk}^{n} + \underline{U} \bullet \nabla \theta_{Lk}^{\#} \right) + \Delta t (r_s^2 \Delta p)_k (R \overline{\omega_k}^{\lambda\phi} / \Pi_k) \left\{ L - q_c^{(L)} + (L_c + L_f)\, q_c^{(F)} \right\} / c_p^2 p_k,$$

(40)

where for simplicity we have included all the terms in (35) in the $U.\nabla$ operator, with the q_T equation being written similarly but without the last thermodynamic term.. The equation for ω_k is given below. The equations for u and v are

$$(\overline{r_s^2 \Delta p}^{\lambda\phi})_k^{n+1} u_k^{n+1} = (\overline{r_s^2 \Delta p}^{\lambda\phi})_k^{n+1} u_k^{n} -$$

(41)

$$\frac{1}{2}\Delta t \left((\overline{\Delta p_k^{n+1} / \Delta p_k^{n}}^{\lambda\phi})\ \underline{U} \bullet \nabla u_k^{n} + \underline{U} \bullet \nabla u_k^{\#} \right) - \Delta t (f_2 + u/r_s)\, \tilde{w}$$

$$(\overline{r_s^2\Delta p}^{\lambda\phi})_k^{n+1} v_k^{n+1} = (\overline{r_s^2\Delta p}^{\lambda\phi})_k^{n+1} v_k^{n} -$$

$$\frac{1}{2}\Delta t \left((\overline{\Delta p_k^{n+1} / \Delta p_k^{n}}^{\lambda\phi})\ \underline{U}\bullet\nabla v_k^n + \underline{U}\bullet\nabla v_k^{\#}\right) - \Delta t(f_1 + v/r_s)\tilde{w}$$

(42)

with all the terms again included in the $U.\nabla$ operator. The approximate mass-weighted vertical

$$\tilde{\omega}_k = -r_s^2 \Delta p R T_s \omega_k / gp \tag{43}$$

where

$$\omega_k = (\dot{\eta}\,\partial p/\partial\eta + \underline{u}\bullet\nabla p + \partial p/\partial t)_k$$

(44)

with $\underline{u}$=(u,v) so that

$$(r_s^2\Delta p)_k \omega_k = \left\{ \frac{1}{2}\left(\overline{E_{k+\frac{1}{2}}}^{\lambda\phi}\ \overline{\Delta p_{k+1/2}^{n+1}}^{\lambda\phi} + \overline{E_{k+1/2}}^{\lambda\phi}\ \overline{\Delta p_{k-1/2}^{n+1}}^{\lambda\phi} \right) + \right.$$

$$\left. \left[\overline{\overline{U_k}^{\lambda\phi\phi}\delta_\lambda\ \overline{p^{n+1}}^{\lambda\phi\lambda}} + \overline{\overline{V_k}^{\lambda\phi\phi}\delta_\phi\ \overline{p^{n+1}}^{\lambda\phi\phi}} \right] / \cos\phi + B_k(\overline{r_s^2\Delta p_k}^{\lambda\phi})^n\ (\overline{p_*^{n+1}}^{\lambda\phi} - \overline{p_*^{n}}^{\lambda\phi})\ /\ \Delta t \right\}$$

(45)

$$\tilde{\omega}_k = -(RT_s / \overline{gp}^{\lambda\phi})_k^{n+1} (r_s^2\Delta p)_k \omega_k$$

(46)

Since the terms involving $\tilde{\omega}_k$ and ω_k are only small correction terms, the more elaborate approximation to ω that would be needed in the thermodynamic equation if T had been used as the model variable is not necessary.

The form of equations (40) to (42) ensure conservation under time differencing.

When the option to halve the timestep at the top level is invoked, the full advection calculation is carried out as above with the terms due to horizontal advection multiplied by $\Delta t/2$ and the term involving ω also multiplied by $\Delta t/2$. The whole calculation is then repeated for the top level only with the vertical advective fluxes set to zero, and the timestep still set to $\Delta t/2$.

3.4 Diffusion and divergence damping

Experience with high resolution limited area models suggests that divergence damping will be needed in forecast as well as assimilation mode for gridlengths below 100 km. The vertical diffusion will also need to be reassessed, this is not covered in this note.

A conservative variable order diffusion scheme is used. The scheme for diffusing a variable X can be written as:

$$(X^{n-1}-X^{n}) \;/\; \Delta t = (-1)^{j-1} D^{j}(X) \tag{47}$$

where for example if $j=3,\; D^3(X) = D(D(D(X)))$, and

$$D(X) = \frac{1}{(r_s^2\Delta p)^{n+1}}\left(\frac{1}{\cos^2\phi} (\overline{K_x(\lambda,\phi,\eta}^{\Phi})\,\delta_\lambda X) + \frac{1}{\cos\phi}\,\delta_\phi(\overline{K_y(\lambda,\phi,\eta}^{\lambda})\;\cos\phi X)\right)$$

where

$$K_y(\lambda,\phi,\eta) = \Delta p^{n+1}K \quad \text{and} \quad K_x(\lambda,\phi,\eta) = K_y(\lambda,\phi,\eta)(\Delta\lambda\cos\phi \;/\; \Delta\phi)^2$$

(48)

The operator D is a conservative second order filter and the diffusion scheme implemented is therefore a conservative $2j$ order filter so for $j=3$ the scheme is sixth order. The choice of which order of diffusion scheme is left to the user and is likely to be resolution dependent. θ_L, u and v are diffused with the same order and coefficent K , both of which can be chosen differently for every level. A different order and coefficient can be used for the diffusion of total water which is inherently a rougher field than the other quantities. The filter removes the same fraction of the shortest resolvable scale at all points of the grid, allowing for the variable resolution. The stability criterion for this scheme is as follows ;

$$\Delta t\left(\frac{4K}{r_s^2\Delta\phi^2}\right)^j \le 1$$

For the diffusion of θ_L and q_T it is necessary to calculate D at the poles. This is done by calculating the ϕ -direction terms in D only as the λ -direction term is zero. These terms are calculated for each meridian and then averaged to give a polar value of D . For u and v diffusion D is evaluated at all points, this involves cross-polar derivatives for the polarmost points. The diffusion acts therefore to remove any gradient in the zonal mean values of the fields between the pole and the surrounding row.

The divergence at level k is defined by equation (30) and is calculated using (u_k, v_k) .

Increments

$$\frac{K_D \Delta t}{(r_s^2 \Delta p)^n \, r_s \cos\phi} \, \delta_\lambda \overline{D}^\phi \quad \text{and} \quad \frac{K_D \Delta t}{r_s (r_s^2 \Delta p)^n} \, \delta_\phi \overline{D}^\lambda$$

(49)

are added to u and v before the advection calculation, where K_D is the divergence damping coefficient, which can be set separately for each model level.

3.5 Fourier Filtering

As in Bell and Dickinson (1987), Fourier filtering is used at high latitudes in the global versions of the model to avoid the need for a very short timestep. The need for filtering in limited area versions of the model is avoided by appropriate choice of the coordinate pole, so that no part of the area lies at a coordinate latitude greater than 45^0 . With any resolution the timestep is chosen so as to minimise the filtering area without introducing too many timesteps per day. In line with other models the filtering area is constrained to go no further than 50^0 with a maximum wind speed of $100 ms^{-1}$ and hence a 10 minute timestep is required for 288 points on a latitude circle. It is also necessary to ensure that global conservation properties are not affected by the filtering. The fields filtered are mass-weighted velocity fields $[\, r_s \, \Delta p \, (u, v) \,]$ during adjustment and after diffusion steps, and the mass-weighted increments to θ_L and q_T due to horizontal advection. Filtering this form of mass-weighted velocity fields before the update to p_* together with filtering (31) removes the need to filter p_* and θ increments after the adjustment steps, so that the conservation proofs of section 4 do not have to consider the effect of filtering. The global means of mass-weighted θ_L, q_T, u and v are therefore not affected by filtering. This strategy also avoids the problem of filtering fields which vary rapidly along a model coordinate surface and preserves the angular momentum principle.

The method of filtering is Fourier damping. This approach is used as fourier chopping was found to interact badly with the model physics leading to problems with noise and "bulls-eyes" at high latitudes. A separate filtering latitude for each hemisphere is calculated. On each pressure row the maximum zonal wind speed u_{MAX} at any level on the two adjacent wind rows is calculated and filtering is then performed for each wave number that satisfies

$$\max\left(\xi_v^2 \frac{\Delta t^2\ \mu^2}{\Delta\lambda^2 \cos\phi^2}\left(\frac{u}{r_s}\right)^2_{\max}, \xi_j^2\ c^2 \frac{\delta t^2\ \mu^2}{\Delta\lambda^2 \cos\phi^2}\left(\frac{1}{r_s}\right)^2_{\max}\right) > 1$$

(50)

For simplicity we set $r_s = a$ for this test, hence

$$\max\left(\xi_v^2 \frac{\Delta t^2\ \mu^2}{a^2 \Delta\lambda^2 \cos\phi^2}\ (u)^2_{\max}, \xi_j^2\ c^2 \frac{\delta t^2}{a^2}\ \frac{\mu^2}{\Delta\lambda^2 \cos\phi^2}\right) > 1$$

where c is the phase speed of the fastest moving gravity wave. μ represents both the 2-D nature of the grid, which is ignored by using just $\Delta\lambda$ in the CFL calculation, and a user supplied safety factor to allow more filtering than the criteria implies. So we write

$$\mu = (1 + (\Delta\lambda\ \cos\phi\ /\ \Delta\phi)^2)^{1/2} + \mu_u$$

where the first term represents the 2-D nature of the grid and the second is the user supplied value currently set to .1, μ_u is not used in calculating the adjustment criteria as the gravity wave speed is constant. The code records the highest stable wave number for each row as well as the first row in each hemisphere which needs filtering. The wind rows are filtered using the same wave number cut-off as the pressure row to poleward of them. ξ_v and ξ_J are the Courant numbers for advection and adjustment respectively with unit wind speed. For the schemes set out in this paper,

$$\xi_v = \sin(kr_s \Delta\lambda \cos\phi)$$

(51)

$$\xi_J = \sin(\frac{1}{2}kr_s \Delta\lambda \cos\phi) \quad (52)$$

where $k = \frac{1}{2}\cos\phi$. This calculation is performed at intervals during the model integration and the length of these intervals can be modified by the user, a typical time-length between calls is 6 hours. Fourier coefficients are calculated by applying the discrete Fourier transform:

$$\tilde{X}(k) \quad (1/N)\sum_{j=0}^{N-1} X(j)\, e^{-2\pi ijk/N}$$

(53)

where N is the number of points round a latitude circle and $-\frac{1}{2}N<k<\frac{1}{2}N+1$. Any coefficient $\tilde{X}(k)$ for which $\xi_v(k)$, $\xi_j(k)$ satisfy (50) is damped by mulitplying the fourier coefficient by the following factor

$$\left(\frac{Maximum\ stable\ wave\ number}{Number\ of\ waves\ on\ a\ row} \right)$$

where the maximum stable wave-number is as calculated as mentioned previously and the number of waves on a row is simply the number of points on a row divided by two. If there are no stable wave-numbers on a row, ie. the maximum stable wave number is zero then the fourier damping becomes fourier chopping
allowing only a mean value increment. The filtered field is then reconstructed using

$$X'(j) = \sum_{k=-\frac{1}{2}N}^{\frac{1}{2}N+1} \tilde{X}(k)\, e^{2\pi ijk?N}$$

(54)

Values of the prognostic variables p_*, θ, q_c, u and v are extracted from a global integration at regular intervals and linearly interpolated to limited area model grid-points in a zone near the limited area model boundary. The limited area model points in the zone are replaced by a weighted mean of limited area values and those interpolated from the global model. It is necessary that the global model run used to generate the boundary values starts from the same initial data as is used to provide interpolated initial data for a limited area model forecast or assimilation.

The values u and v are stored a half grid-length from the poles. These values are updated by adding the correct flux to each u and v value and then calling the polar_uv routine. This routine calculates a mean u and v as well as a local cartesian value (u_0, v_0) obtained by

$$\begin{aligned} u &= u_0\cos\lambda + v_0\sin\lambda \\ v &= v_0\cos\lambda + u_0\sin\lambda \end{aligned} \tag{55}$$

The routine takes u and v at the row adjacent to the polar row as the values to work on. The values stored at the polar row are one-third the mean of u and v plus the local cartesian wind on this row. In each adjustment step the meridional fluxes are

$$\frac{1}{\cos\phi}\ \delta\phi\left(v_m \overline{r_s \Delta p_m}^{\lambda\phi}\ \cos\phi\right) \tag{56}$$

which are summed around a latitude circle then averaged to form a polar value of D, which is then used to update p_* and θ as in equations (27-29). The divergence damping increment to u given by (49) is averaged to give a polar increment.

In the advection step the fluxes of θ_L and q_T into the area contained within a half grid length from the pole are calculated. These fluxes are then averaged to give a polar increment. No east-west fluxes are calculated for the polar value. For u and v no advective increment is calculated for the polarmost row. The increments for all other rows are calculated normally and added to the field. The polarmost values after advection are then obtained by applying the polar boundary condition described previously.

3.7 Recalculation of primary model variables

Negative values of moisture are removed by summing the mass-weighted negative values in a layer and setting them to zero. Then summing all the mass-weighted positive values and rescale them by

$$q_T{}' = q_T\left(1 + \frac{\sum negative}{\sum positive}\right)$$

This is conservative, with the mass-weighting defined as $r_s^2 \Delta p \cos\phi$, since

$$\sum\nolimits_{all\ points} q_T{}' = \sum\nolimits_{all\ points} q_T\left(1 + \frac{\sum negative}{\sum positive}\right)$$

$$\sum\nolimits_{pos.points} q_T\left(1+\frac{\sum negative}{\sum positive}\right) + \sum\nolimits_{neg.points} q_T\left(1 + \sum \frac{neg.points}{pos.points}\right)$$
$$= \sum\nolimits_{pos.points} q_T + \sum\nolimits_{neg.points} q_T$$

$$\sum\nolimits_{all\ points} q_T$$

In the event of the negative values in a layer outweighing the positive values, there is an option to allow a run to continue without conservation by omitting the rescaling.

The primary model variables are recovered from θ_L and q_T by setting

$$T_{LK} = \Pi_K \theta_{Lk} \tag{57}$$

Call the cloud scheme described in Unified Model Documentation paper no. 29. This calculates

$$q_c^{(1)},\ q_c^{(F)} = (q_c^{(1)}, q_c^{(f)})\,(T_L, q_T, p_*)$$

(58)

and returns values of q, $q_c^{(1)}$, $q_c^{(f)}$, T and cloud fraction. Finally set

$$\theta = T / \Pi \tag{59}$$

4. CONSERVATION PROPERTIES

4.1 Angular momentum conservation

The requirement is that the pressure gradient term can only change the angular momentum through the surface torque. This means that we must be able to write the approximation to the pressure gradient term in the model which is

$$\sum_{m=1}^{TOP} \left(\frac{\vartheta \Phi_m}{\vartheta \lambda} + C_p (\theta_v + \mu\theta_s) \frac{\vartheta \Pi_m}{\vartheta \lambda} \right) \Delta p_m \tag{62}$$

in the form

$$\frac{\vartheta}{\vartheta \lambda} \left(\sum_{m=1}^{TOP} \Phi_m \Delta p_m \right) - \Phi_* \frac{\vartheta p_*}{\vartheta \lambda} \tag{63}$$

The first term in (63) integrates to zero and the second integrates to the surface torque. This requirement determines how the terms in (62) have to be calculated at level m, as in Simmons and Strűfing (1981). Cancelling terms gives the requirement

$$\sum_{m=1}^{TOP} C_p (\theta_v + \mu\theta_s) \frac{\vartheta \Pi_m}{\vartheta \lambda} \Delta p_m = \sum_{m=1}^{TOP} \phi_m \frac{\vartheta}{\vartheta \lambda} (\Delta p_m) - \phi_* \frac{\vartheta p_*}{\vartheta \lambda}$$

(64)

We now substitute for Φ in the right hand side of (64) to establish the required form of approximation to the left hand side. Write (26) as

$$\Phi_k = \Phi_* - \sum_{m=1}^{k-1} C_p (\theta_{vm} + \mu\theta_s) \Delta\Pi_m + \alpha_k C_p \Pi_k (\theta_v + \mu\theta_s)_k \tag{65}$$

Then the right hand side of (64) becomes

$$\sum_{m=1}^{TOP} \left\{ \alpha_m C_p \Pi_m (\theta_{vm} + \mu\theta_s) - \Phi_{m-\frac{1}{2}} \right\} \frac{\vartheta}{\vartheta \lambda} \Delta p_m - \Phi_* \frac{\vartheta p_*}{\vartheta \lambda} \tag{66}$$

which is

$$\sum_{m=1}^{TOP} \left\{ \alpha_m C_p \Pi_m (\theta_v + \mu\theta_s)_m \frac{\vartheta}{\vartheta\lambda} \Delta p_m \right\} - \Sigma_{m-1}^{TOP} \Sigma_{j=1}^{m-1} C_p (\theta_v + \mu\theta_s)_j \, \Delta\Pi_j \frac{\vartheta}{\vartheta\lambda} \Delta p_m \quad (67)$$

The double sum in (67) can be written

$$- \sum_{m=1}^{TOP} C_p (\theta_v + \mu\theta_s)_m \, \Delta\Pi_m \, \Sigma_{j=m+1}^{TOP} \frac{\vartheta}{\vartheta\lambda} \Delta p_j \quad (68)$$

where we have used the convention that

$$\Sigma_{j=r}^{r-1} = 0 \quad (69)$$

The second sum is now just $\vartheta / \vartheta\lambda \, (p_m + \frac{1}{2})$. (67) can therefore be written as:

$$\sum_{m=1}^{TOP} \frac{1}{\Delta p_m} \left[C_p \alpha_m \Pi_m (\theta_v + \mu\theta_s)_m \frac{\vartheta}{\vartheta\lambda} \Delta p_m + C_p (\theta_v + \mu\theta_s)_m \Delta\Pi_m \frac{\vartheta}{\vartheta\lambda} p_{(m+1/2)} \right] \Delta p_m$$

(70)

Choose α_m such that

$$\alpha_m \Pi_m = \Pi_{m-\frac{1}{2}} - \frac{\Delta (\Pi p)_m}{(\kappa+1) \Delta p} \quad (71)$$

Then the summand in (70) becomes

$$\frac{C_p (\theta + v + \mu\theta_s)_m}{\Delta p_m} \left[\Pi_{m-\frac{1}{2}} \frac{\vartheta}{\vartheta\lambda} \Delta p_m + \frac{\Delta (\Pi p)_m}{(\kappa+1) \Delta p_m} \frac{\vartheta}{\vartheta\lambda} \Delta p_m - \Delta\Pi_m \frac{\vartheta}{\vartheta\lambda} p_{m+\frac{1}{2}} \right]$$

$$= \frac{C_p (\theta_v + \mu\theta_s)_m}{\Delta p_m} \left[\Pi_{m+\frac{1}{2}} \frac{\vartheta}{\vartheta\lambda} p_{m+\frac{1}{2}} - \Pi_{m-\frac{1}{2}} \frac{\vartheta}{\vartheta\lambda} p_{m-\frac{1}{2}} - \frac{\Delta (\Pi p)_m}{(\kappa+1) \Delta p_m} \frac{\vartheta}{\vartheta\lambda} \Delta p_m \right] \quad (72)$$

Using the definition of Π in terms of p , (72) reduces to

$$\frac{C_p (\theta_v + \mu\theta_s)_m}{(\kappa+1)} \left[\frac{\vartheta}{\vartheta\lambda} \left(\frac{\Delta (\Pi p)}{\Delta p_m} \right) \right] \quad (73)$$

The expression (73) is then approximated by spatial finite differences and used to approximate the term on the left hand side of (64) in the equation of motion. The above argument can be carried through in finite differences, provided that the approximation used is

$$\frac{C_p\,(\overline{\theta_v+\mu\theta_s}^{\lambda})_m}{(\kappa+1)}\left[\delta_\lambda\left(\frac{\Delta\,(\Pi p)_m}{\Delta p_m}\right)\right] \tag{74}$$

as used in equations (23) and (24).

4.2 Conservation of first moments

The requirement is that the global mass-weighted mean of all advected quantities is conserved. The proof is written out only for meridional advection of θ and u , since this covers all the possible staggerings of variables that occur in the other cases. Combining (29), (30), (31) and (35) gives the continuity equation in the form:

$$E_{k+\frac{1}{2}} = -\left(\frac{r_s^2(p)}{r_s^2(p_*)}\;\frac{\vartheta p}{\vartheta p_*}\right)\sum_{m=1}^{TOP} D_m + \sum_{m=k+1}^{TOP} D_m \tag{75}$$

where

$$D_m = \frac{1}{\cos\phi}\delta_\phi \overline{V_m}^{\lambda} \tag{76}$$

A simple second order forward update of θ by meridional advection, and advection by the vertical motion associated with the meridional motion, is given by

$$(r_s^2\Delta p_k)^{+}\theta_k^{+} = r_s^2\Delta p_k^{+}\theta_k - \Delta t\left[\frac{1}{\cos\phi}\overline{\overline{V_k}^{\lambda}\delta_\Phi\theta_k}^{\Phi} +\right.$$
$$\left.\frac{1}{2}\left\{E_{k+\frac{1}{2}}(\theta_{k+1}-\theta_k) + E_{k-\frac{1}{2}}(\theta_k-\theta_{k-1})\right\}\right]$$
$$-\theta_k(p_*^{n+1}-p_*^{n})\;/\;4\;\;(B_{k+1/2}(3r_s(p_k) + r_s(p_{k+1}))(r_s(p_k) - r_s(p_{k+1}) -$$
$$B_{k-1/2}(3r_s(p_k) + r_s(p_{k-1}))(p_k) - r_s(p_{k-1})))$$

(77)

with the extra terms needed for conservation. The update of p_* can be written

$$p_*^+ = p_* + \frac{\Delta t}{r_s^2(p_*)} \sum_{m=1}^{TOP} D_m \tag{78}$$

because of the definition of V_m as the average over the adjustment steps.

Then use (21) to give

$$\Delta p_k^+ = \Delta A_k + \Delta B_k p_*^+ \tag{79}$$

Equation (20) can be used to rewrite (75) as

$$E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}} = -B_{k+\frac{1}{2}}\ r_s^2(p_{k+\frac{1}{2}})\ (p_*^+ - p_*)\ /\ \Delta t\ +$$

$$B_{k-\frac{1}{2}}\ r_s^2(p_{k-\frac{1}{2}})\ (p_*^+ - p_*)\ /\ \Delta t - D_k$$

(80)

If we set $r_s(p_{k+1/2}) = .5^*\ (r_s(p_{k+1}) + r_s(p_k))$, and apply some simple algebra to ensure obtaining the first term on the right-hand side of the following equation, then we can write (80) as

$$\begin{aligned} E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}} = & -\Delta B_k\ r_s^2(p_k)\ (p_*^+ - p_*)\ /\ \Delta t - D_k + \\ & .25*B_{k+\frac{1}{2}}\ (p_*^+ - p_*)\ /\ \Delta t\ (3r_s(p_k) + r_s(p_{k+1}))\ (r_s(p_k) - r_s(p_{k+1})) \\ & -.25*B_{k-\frac{1}{2}}\ (p_*^+ - p_*)\ /\ \Delta t\ (3r_s(p_k) + r_s(p_{k-1}))\ (r_s(p_{k-1})) \end{aligned}$$

A simpler and neater form can be obtained by approximating $r_s^2(p_{k+1/2})$ as

$$r_s^2(p_{k+1/2}) = .5 * (r_s^2(p_k) + r_s^2(p_{k+1}))$$

but this would be inconsistent with the way all other variables are approximated at half levels.Taking this new form of (80) and multiplying by θ_k then adding to (77) gives

$$(\Delta A_k+\Delta B_k p_*^+)(r_s^2\theta_k^+-r_s^2\theta_k) + \theta_k r_s^2(p_*^+ - p_*)\Delta B_k =$$

$$-\Delta t \left[\frac{1}{\cos\phi} \{\overline{\overline{V_k}^{\lambda}\delta_\phi\theta_k}^{\phi} + \theta_k\overline{\delta_\phi V_k}^{\lambda}\} + \right.$$

$$\left.\frac{1}{2}\left\{E_{k+\frac{1}{2}}(\theta_{k+1} - \theta_k) + E_{k-\frac{1}{2}}(\theta_k - \theta_{k-1}) + 2\theta_k(E_{k+\frac{1}{2}} - E_{k-\frac{1}{2}}\right\}\right]$$

(81)

This reduces to

$$(r_s^2\Delta p_k\theta_k)^+ - r_s^2\Delta p_k\theta_k = -\Delta t\left[\frac{1}{\cos\phi}\ \delta_\phi\overline{V_k}^{\lambda}\ \overline{\theta_k}^{\phi} + \right.$$

$$\left.\frac{1}{2}\left\{E_{k+\frac{1}{2}}(\theta_{k+1} + \theta_k) - (\theta_k + \theta_{k-1})\right\}\right]$$

(82)

which gives the desired conservation integral when multiplied by $\cos\phi$ and integrated over ϕ .

The update of u by meridional advection and advection by the associated part of the vertical motion is given by

$$(r_s^2\Delta p_k)_{uk}^{++} = r_s^2\Delta p_k u_k - \Delta t\left[\left(\frac{1}{\cos\phi}\right)\overline{\overline{\overline{V_k}^{\lambda}\phi}^{\lambda}\ \delta_\phi\ u_k}^{\phi}\right.$$

$$\left.\overline{E_{k-\frac{1}{2}}}^{\lambda\phi}\ (u_k - u_{k-1})\}\right]$$

$$- u_k(\overline{p_*^{n+1}}^{\lambda\phi} - \overline{p_*^{n}}^{\lambda\phi})\ /\ 4\ (B_{k+1/2}\ (3\overline{r_s(p_k}^{\lambda\phi}) + (\overline{r_s(p_{k+1}}^{\lambda\phi}))(\overline{r_s(p_k}^{\lambda\phi}) - \overline{r_s(p_{k+1}}^{\lambda\phi}))$$

$$B_{k-1/2}\ (\overline{3r_s(p_k}^{\lambda\phi}) + \overline{r_s(p_{k-1}}^{\lambda\phi}))\ (\overline{r_s(p_k}^{\lambda\phi}) - \overline{r_s(p_{k-1}}^{\lambda\phi})))$$

(83)

Multiplying the modified form of $\overline{(80)}^{\lambda\phi}$ by u_k and adding gives

$$\overline{(r_s^2\Delta p^+)_k}^{\lambda\phi}u_k^+ - \overline{(r_s\Delta p)_k}^{\lambda\phi}u_k = -\Delta t\left[\left(\frac{1}{\cos\phi}\right)\{\overline{\overline{V_k}^{\lambda\phi\lambda}\delta_\phi u_k}^{\phi} + u_k\overline{\delta_\phi V_k}^{\lambda\phi}\} + \frac{1}{2}\left\{\overline{E_{k+\frac{1}{2}}}^{\lambda\phi}(u_{k+1} - u_k) + (\overline{E_{k-\frac{1}{2}}}^{\lambda\phi}(u_k - u_{k-1}) + 2u_k(\overline{E_{k+\frac{1}{2}}}^{\lambda\phi} - \overline{E_{k-\frac{1}{2}}}^{\lambda\phi})\right\}\right] - \frac{1}{2}\Delta t\left\{\overline{E_{k+\frac{1}{2}}}^{\lambda\phi}(u_{k+1} - u_k) + \overline{E_{k-\frac{1}{2}}}^{\lambda\phi}(u_k - u_{k-1})\right\}$$

(84)

The right hand side of (84) reduces to

$$-\Delta t\left[\left(\frac{1}{\cos\phi}\right)\left\{\delta_\phi(\overline{\overline{V_k}^{\lambda\phi\lambda}}\ \overline{u_k}^{\phi})\right\} + \frac{1}{2}\left\{\overline{E_{k+\frac{1}{2}}}^{\lambda\phi}(u_{k+1}+u_k) - (\overline{E_{k-\frac{1}{2}}}^{\lambda\phi}(u_k+u_{k+1})\right\}\right]$$

(85)

This is in conservation form.

Now consider the fourth order terms in (34) to (37). Conservation cannot be achieved if v is a function of ξ, as may be necessary to avoid reducing the timestep. Suppose that v is a constant. The terms

$$\overline{(1+v)\overline{U_k}^{\phi}\delta_\lambda\theta_k}^{\lambda} - \overline{v\overline{U_k}^{\phi}\delta_\lambda\phi_k}^{3\lambda}$$

can be expanded as

$$(1 + v)\ \overline{U_k}^{\phi}\ (\lambda+\frac{1}{2}\Delta\lambda)(\theta_k(\lambda+\lambda\Delta)-\theta_k(\lambda)) -$$

$$v\overline{U_k}^{\phi}(\lambda+3/2\Delta\lambda)(\theta_k(\lambda+2\Delta\lambda)-\theta_k(\lambda+\Delta\lambda)$$

(86)

with symmetrical terms in $-\Delta\lambda$. These terms cancel with contributions from $\theta_k^+(\lambda+\Delta\lambda)$ and $\theta_k^+(\lambda-\Delta\lambda)$ when p_*^+ is summed over λ to give the required conservation. At the poles, the omission of the factor $(1+v)$ in the fluxes summed to give the polar increment is cancelled by the omission of the second term in (86) when incrementing the points adjacent to the pole.

4.3 Conservation of second moments

It can be shown that the integral of the second moment of any of the primary variables is not conserved using the second order accurate approximation to the advection terms unless we take r_s as constant.We illustrate this by looking at θ conservation. To work through the analysis multiply (80) by θ_k^2 and add to (77) multiplied by $2\theta_k$ in a similar way to the proof of conservation of first moments. The left hand side is then a discrete approximation to

$$\Delta p_k \frac{\partial}{\partial t}(r_s^2\theta_k^2) + r_s^2\theta_k^2 \frac{\partial}{\partial t}\Delta p_k$$

However, it cannot be written as exact conservation of $r_s^2\Delta p_k\theta_k^2$. The right hand side becomes

$$-\Delta t\left[\frac{1}{\cos\phi}\delta_\phi \overline{V_k}^\lambda \left(2(\overline{\theta_k}^\phi)^2 - \overline{\theta_k^2}^\phi\right) + \theta_k\theta_{k+1} E_{k+\frac{1}{2}} - \theta_k\theta_{k-1} E_{k-\frac{1}{2}}\right]$$

$$-\theta_k^2(p_*^{n+1}-p_*^n) / 4(B_{k+1/2}(3r_s(p_k) + r_s(p_{k+1}))(r_s(p_k)-r_s(p_{k+1})) - B_{k-1/2}(3r_s(p_k) + r_s(p_{k-1}))(r_s(p_k)-r_s(p_{k-1})))$$

The terms inside the square bracket are in conservation form but the remaining terms are not conservative. Should it be possible to remove these extra terms then the scheme would conserve but would not have quadratic conservation with the fourth order terms included unless the E_s *'s* are redefined (Fisher, private communication). The resulting scheme is rather less accurate because it uses a broader stencil of gridpoints. If we remove the dependence of the radius of Earth on pressure then we get second moment conservation. However removing this dependence would be undesirable if we were still to keep the other small terms suggested by White and Bromley.

4.4 Global mass and energy corrections

Though the total mass of the atmosphere is conserved by the finite difference scheme, in very long integrations the loss of conservation due to computer arithmetic may be significant. The total mass

$$\int\int r_s^2(p_*)\,p_*\cos\phi\ d\lambda d\phi \tag{93}$$

is therefore computed at the start of an integration and reset to this value at intervals by adding a globally uniform correction to p_* .

Total energy is not conserved by the dynamical equations because of the presence of diffusion terms, and further energy loss results from Fourier filtering. A globally uniform temperature correction is made by inferring a total energy change from fluxes computed in the physics routines. Since the method of calculation depends on the physics, it is documented separately in Unified Model Documentation Paper no. 18.

5. SUMMARY

This note has outlined a conservative split-explicit finite difference scheme on a B grid. There are some choices to be made. Experiments should be conducted to see if climate integrations perform better with a second order scheme or the fourth order scheme proposed by Fisher (not coded at present), giving quadratic conservation, or the fourth order scheme given here with constant *v*, giving more accuracy. The scheme has intentionally been written using an approximation to the Lagrangian derivative of momentum in the momentum equation, rather than using the alternative vorticity/ energy form. The latter form can lead to spurious sources or sinks of momentum, though it allows enstrophy to be conserved. Moreover, if the solutions are not smooth, it is more important to treat the momentum correctly than the vorticity.

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APPENDIX 1 – STANDARD CONFIGURATIONS OF THE MODEL

1. Vertical resolution

The Climate, Global Forecast and Limited Area Forecast use the following set of 19 levels.

Level	$A_{K+1/2}$	$B_{K+1/2}$	η
19.5	50.0	0.0	0.0005
18.5	1000.0	0.0	0.01
17.5	2000.0	0.0	0.02
16.5	4000.0	0.0	0.04
15.5	7176.0	0.003239	0.075
14.5	10652.1	0.018478	0.125
13.5	12997.5	0.045024	0.175
12.5	14342.7	0.081572	0.225
11.5	14818.3	0.126816	0.275
10.5	14555.1	0.179448	0.325
9.5	13447.6	0.250523	0.385
8.5	11175.3	0.348246	0.46
7.5	7727.9	0.472720	0.55
6.5	3852.2	0.611477	0.65
5.5	939.0	0.740609	0.75
4.5	0.0	0.835	0.835
3.5	0.0	0.905	0.905
2.5	0.0	0.956	0.956
1.5	0.0	0.994	0.994
0.5	0.0	1.0	1.0

Using the interpolation scheme defined following eq. (22) gives A_k, B_k values:

k	A_k	B_k	η_k
19	460.6	0.0	0.004606
18	1479.7	0.0	0.014797
17	2959.4	0.0	0.029594
16	5529.4	0.001560	0.056854
15	8861.7	0.010630	0.099247
14	11801.4	0.031487	0.149501
13	13660.1	0.063026	0.199627
12	14577.7	0.103925	0.249702
11	14688.1	0.152871	0.299752
10	14007.0	0.214628	0.354698
9	12323.5	0.298868	0.422103
8	9469.9	0.409823	0.504522
7	5809.3	0.541410	0.599503
6	2408.0	0.675494	0.699574
5	472.5	0.787503	0.792229
4	0.0	0.869832	0.869834
3	0.0	0.930417	0.930417
2	0.0	0.974956	0.974956
1	0.0	0.996999	0.996999

The following 30 levels are used in the operational Mesoscale model.

Level	$A_{k+1/2}$	$B_{k+1/2}$	η
30.5	50.0	0.000000	0.0005
29.5	1000.0	0.000000	0.010
28.5	2000.0	0.000000	0.020
27.5	4000.0	0.000000	0.040
26.5	7176.0	0.003240	0.075
25.5	10652.2	0.018478	0.125
24.5	12997.5	0.045025	0.175
23.5	14342.7	0.081573	0.225
22.5	14818.4	0.126816	0.275
21.5	14555.1	0.179449	0.325
20.5	13447.7	0.250523	0.385
19.5	11689.9	0.328101	0.445
18.5	9700.1	0.402999	0.500
17.5	7527.4	0.479726	0.555
16.5	5345.5	0.556545	0.610
15.5	3500.4	0.624996	0.660
14.5	2064.1	0.684359	0.705
13.5	1045.7	0.734543	0.745
12.5	409.0	0.775910	0.780
11.5	87.8	0.809122	0.810
10.5	0.0	0.835000	0.835
9.5	0.0	0.858000	0.858
8.5	0.0	0.880000	0.880
7.5	0.0	0.901000	0.901
6.5	0.0	0.921000	0.921
5.5	0.0	0.940000	0.940
4.5	0.0	0.957000	0.957
3.5	0.0	0.972000	0.972
2.5	0.0	0.985000	0.985
1.5	0.0	0.994000	0.994
0.5	0.0	1.000000	1.000

Using the interpolation scheme defined following eq. (22) gives A_k, B_k values

k	A_k	B_k	η_k
30	460.6	0.000000	0.004606
29	1479.7	0.000000	0.014797
28	2959.4	0.000000	0.029594
27	5529.4	0.001560	0.056854
26	8861.7	0.010630	0.099247
25	11801.4	0.031487	0.149501
24	13660.1	0.063026	0.199627
23	14577.7	0.103925	0.249702
22	14688.1	0.152871	0.299752
21	14007.0	0.214628	0.354698
20	12576.4	0.288978	0.414742
19	10701.9	0.365290	0.472309
18	8620.5	0.441124	0.527329
17	6442.6	0.517920	0.582345
16	4427.3	0.590610	0.634883
15	2785.0	0.654561	0.682412
14	1556.6	0.709369	0.724934
13	728.2	0.755170	0.762452
12	248.7	0.792479	0.794966
11	44.0	0.822038	0.822477
10	0.0	0.846481	0.846481
9	0.0	0.868983	0.868983
8	0.0	0.890485	0.890485
7	0.0	0.910987	0.910987
6	0.0	0.930488	0.930488
5	0.0	0.948491	0.948491

4	0.0	0.964493	0.964493
3	0.0	0.978495	0.978495
2	0.0	0.989498	0.989498
1	0.0	0.996999	0.996999

The following 42 levels are used in stratosperic studies. Other levels eg: 49 or 31 are also used but not included here.

Level	$A_{K+1/2}$	$B_{k+1/2}$	η
42.5	25.1	0.0	0.000251
41.5	31.6	0.0	0.000316
40.5	39.8	0.0	0.000398
39.5	50.1	0.0	0.000501
38.5	63.1	0.0	0.000631
37.5	79.4	0.0	0.000794
36.5	100.0	0.0	0.001000
35.5	125.9	0.0	0.001259
34.5	158.5	0.0	0.001585
33.5	199.5	0.0	0.001995
32.5	251.2	0.0	0.002512
31.5	316.2	0.0	0.003162
30.5	398.1	0.0	0.003981
29.5	501.2	0.0	0.005012
28.5	631.0	0.0	0.006310
27.5	794.3	0.0	0.007943
26.5	1000.0	0.0	0.010000
25.5	1258.9	0.0	0.012589
24.4	1584.9	0.0	0.015849
23.5	1995.3	0.0	0.019953
22.5	2511.9	0.0	0.025119
21.5	3162.3	0.0	0.031623
20.5	3981.1	0.0	0.039811
19.5	5011.9	0.0	0.050119
18.5	6263.0	0.000466	0.063096
17.5	7708.0	0.002353	0.079435
16.5	9328.0	0.006720	0.100000
15.5	11041.8	0.015082	0.125500
14.5	12733.8	0.029662	0.157000
13.5	14226.1	0.053739	0.196000
12.5	15112.7	0.083873	0.235000
11.5	15461.1	0.120389	0.275000
10.5	15204.1	0.172959	0.325000
9.5	14061.3	0.244387	0.385000
8.5	11695.5	0.343045	0.460000
7.5	8093.5	0.469065	0.550000
6.5	4036.8	0.609632	0.650000
5.5	984.4	0.740156	0.750000
4.5	0.0	0.835000	0.835000
3.5	0.0	0.905000	0.905000
2.5	0.0	0.956000	0.956000
1.5	0.0	0.994000	0.994000
0.5	0.0	1.000000	1.000000

Using the interpolation scheme defined following eq. (22) gives A_k, B_k values

k	A_k	B_k	η_k
42	28.3	0.0	0.000283
41	35.7	0.0	0.000357
40	44.9	0.0	0.000449
39	56.5	0.0	0.000565
38	71.2	0.0	0.000712
37	89.6	0.0	0.000896
36	112.7	0.0	0.001127
35	142.0	0.0	0.001420
34	178.7	0.0	0.001787
33	225.0	0.0	0.002250
32	283.3	0.0	0.002832
31	356.6	0.0	0.003566
30	448.9	0.0	0.004489
29	565.2	0.0	0.005652
28	711.5	0.0	0.007115
27	895.8	0.0	0.008958
26	1127.7	0.0	0.011277
25	1419.7	0.0	0.014197
24	1787.3	0.0	0.017873
23	2250.0	0.0	0.022500
22	2832.6	0.0	0.028326
21	3566.1	0.0	0.035661
20	4489.4	0.0	0.044894
19	5628.9	0.000230	0.056519
18	6975.6	0.001397	0.071153
17	8506.9	0.004507	0.089576
16	10173.3	0.010845	0.112578
15	11876.5	0.022275	0.141040
14	13470.1	0.041542	0.176243
13	14664.6	0.068644	0.215290
12	15285.3	0.101960	0.254813
11	15333.9	0.146413	0.299752
10	14638.5	0.208313	0.354698
9	12890.9	0.293194	0.422103
8	9913.6	0.405386	0.504532
7	6085.3	0.538650	0.599503
6	2523.6	0.674338	0.699574
5	495.4	0.787275	0.792229
4	0.0	0.869832	0.869832
3	0.0	0.930416	0.930416
2	0.0	0.974955	0.975955
1	0.0	0.996999	0.996999

2. *Horizontal resolution and model area*

The standard horizontal resolutions will be:

	points round latitude circle	points from North Pole to South Pole inclusive	Timestep (minutes)
Operational forecast	288	217	10
Long range forecast	192	145	15
Seasonal forecast	144	109	20
Climate studies	96	73	30
Stratosphere studies	96	73	20
Limited Area Forecast	229	132	5
Mesoscale Model	92	92	1.5

Limited area model

The operational limited area forecast area is based on a coordinate pole at 30°N, 160°E. The corners of the area are approximately in actual latitude and longitude: (45.1N, 117.4W; 45.9N, 77.1E; 11.2N, 62.0W; 11.7N, 21.4E). Relative to the coordinate pole the corners are at (25.66N, 50.89W; 25.66N, 50.00E; 32.31S, 50.89W; 32.31S, 50.00E). The gridlength is 0.442° in each direction (approximately 50km), giving 229x132 points. The boundary updating for the limited area model uses a zone 4 points in width, with replacement weights starting from the boundary of (1, .75, .5, .25).

Mesoscale Model

The operational Mesoscale model is based on a co-ordinate pole at 37.5°N, 177.5°E. The corners of the area are approximately in actual latitude and longitude: (60.1N, 16.6W; 60.2N, 10.7E; 46.6N, 12.7W; 46.7N, 7.1E). Relative to the coordinate pole the corners are at (8.25N, 7.05W; 8.25N, 6.6E; 5.4S, 7.05W; 5.4S, 6.6E). The gridlength is 0.150° in each direction (approximately 16.8km), giving 92x92 points. The boundary updating for the Mesoscale model uses a zone 4 points in width, with replacement weights starting from the boundary of (1, .75, .5, .25).

APPENDIX 2

°

STANDARD ATMOSPHERE

The standard atmosphere consists of an extension of the International Standard Atmosphere (ISA) upwards using reference mesosphere. Between specified pressure values, the standard atmosphere is either isothermal or obeys the following equation:

$$T(p) = T(p_b)(p_b/p)^{(RL/g)} \quad \text{(A1)}$$

where p_b is the pressure at the bottom of a layer and L is a lapse rate in K/m. Values of T, p and L in the various layers are given in Table 1. All the values are for the ISA except for the top level. For pressures greater than 101325, equation (A1) is applied using p_b =101325 and L=-0.0065.

Table 1

P_b (pa)	$T(p_b)$ (K)	L(K/m)
101325	288.15	
		-0.00065
22632	216.65	
		0
5475	216.65	
		0.0010
868	228.65	
		0.0028
111	270.65	
		0
75	270.65	
		-0.0028
.0001	89.309	0

As no standard atmosphere is defined above 75pa, values were compiled by examining satellite data given by Barnet and Corney (1985). This contains average temperatures for each month at fixed pressure levels averaged in 10° latitude bands. The band which agreed most closely with the ISA at 100 pa was selected, giving the values shown in Table 2. The values were then averaged over the year and a typical lapse rate calculated by using equation (A1), the point at the top of the ISA (75pa), and the average temperature at the top data point. This gives a lapse rate of -0.0028°K/m. The departure of the average temperature values at each pressure level from the one calculated using this lapse rate and equation (A1) are shown in the 'error' row in Table 2.

Table 2

Month/latitude band	Pressure level(pa)								
	1.03	1.69	2.79	4.60	7.58	12.50	20.61	33.98	56.03
Jan 50S	178.7	188.9	199.8	210.7	221.4	232.5	245.1	257.0	267.1
Feb 0N	205.4	206.9	209.9	214.7	222.4	232.6	244.6	257.7	268.1
Mar 0N	209.0	208.8	210.5	214.5	221.2	230.5	242.0	254.5	266.1
Apr 50N	199.8	209.0	217.7	224.1	230.2	236.4	243.1	252.8	263.5
May 50N	185.2	195.9	206.4	215.5	224.4	234.0	243.8	254.6	265.9
Jun 50N	175.8	186.5	197.5	208.2	219.4	231.5	244.0	256.2	267.0
Jul 50N	175.9	185.9	196.6	207.4	218.2	229.4	242.0	253.8	263.7
Aug 70N	174.0	184.9	197.5	210.9	224.9	238.4	249.7	258.7	266.3
Sep 70S	214.9	219.0	222.9	227.1	232.0	239.2	249.9	261.7	268.3
Oct 60S	201.6	211.9	220.8	227.6	232.9	238.2	244.5	252.7	261.8
Nov 50S	186.8	191.4	207.8	216.8	225.3	234.6	244.1	254.8	265.5
Dec 40S	188.5	195.4	202.8	210.5	219.9	231.1	242.8	254.9	265.5
Average	191.3	198.9	207.5	215.6	224.4	234.0	244.3	255.8	265.8
Error	0	0.23	−0.11	0.36	0.47	0.17	−0.45	−1.9	−1.4

Coding implementation

Calculating T(p) from equation (A1) using full exponenatiation is very expensive. In the code available are two cheaper alternatives [the selection of which is controlled by *DEF LINEARTS].

a) [*DEF LINEARTS enabled]: A simple and very cheap linear approximation to T(p)is calculated for all values of p.

$$T(p) = Ap + B$$

where A and B are chosen so that T is correct at p=22632 pa and 101325 pa.

b) [*DEF LINEARTS disabled]: The code determines which layer p is in and then applies equation A1 exactly, except that the exponentiation is replaced by a Taylor series expanded about the mid-pressure value for that layer and terminated at the sixth term.

APPENDIX 3

Dynamical Fluxes from the Unified Model.

The following quantities can be requested as diagnostics from the Unified Model. They are available at all points on all model levels and are produced at the end of the model's adjustment steps. Application of suitable finite-differencing will then give a reasonable approximation to the model advective increments. Note that for most of these quantities there is explicit model flux. Also, the advecting velocity in the model is not the wind field at the end of the adjustment step but the mean wind field during the adjustment step. These values are usually similar but may vary significantly in areas of rapid change. We denote the mean wind field during the advection step by (u_a,v_a,w_a). The diagnostics available at present and the grid-location where calculated (see figs 1a,1b for explanation) are,

Diagnostic	Grid-location
u_a p T	$p_{I+1/2,j}$ points.
$v_a \Delta p$ T	$p_{i,j+1/2}$ points.
$u_a \Delta p$ T_L	$p_{I+1/2,j}$ points.
$v_a \Delta p$ T_L	$p_{i,j+1/2}$ points.
$u_a \Delta p$ Q	$p_{I+1/2,j}$ points.
$v_a \Delta p$ Q	$p_{i,j+1/2}$ points.
$u_a \Delta p$ Q_T	$p_{I+1/2,j}$ points.
$v_a \Delta p$ Q_T	$p_{i,j+1/2}$ points.
$u_a \Delta p$ ϕ	$p_{I+1/2,j}$ points.
$v_a \Delta p$ ϕ	$p_{i,j+1/2}$ points.
$u_a \Delta p$	$u_{i,j}$ points.(mass-flux)
$v_a \Delta p$	$u_{i,j}$ points.(mass-flux)
$u_a \Delta p$ u	$u_{i,j}$ points.
$u_a \Delta p$ v	$u_{i,j}$ points.
$v_a \Delta p$ v	$u_{i,j}$ points.
$v_a \Delta p$ u	$u_{i,j}$ points.
$u_a \Delta p$ $(C_pT_L + L_cQ_T + \phi)$	$p_{I+1/2,j}$ points.(moist static energy flux)
$v_a \Delta p$ $(C_pT_L + L_cQ_T + \phi)$	$p_{i,j+1/2}$ points.(moist static energy flux)

where ϕ = gz takes the value calculated in the adjustment step.

Figure 1.a

$p_{i,j}$	$p_{i+1/2,j}$	$p_{i+1,j}$
$p_{i,j+1/2}$	$u_{i,j}$	$p_{i+1,j+1/2}$
$p_{i,j+1}$	$p_{i+1/2,j+1}$	$p_{i+1,j+1}$

Fig 1a., above, represents the usual model variable staggering where p_* ,θ,q are held at the full p points and u,v are held at the full u points. Fig 1b.,below, shows the representation with respect to Arakawa B and C grids. The point $p_{i+1/2,j}$ is the point where the u component of velocity would be held

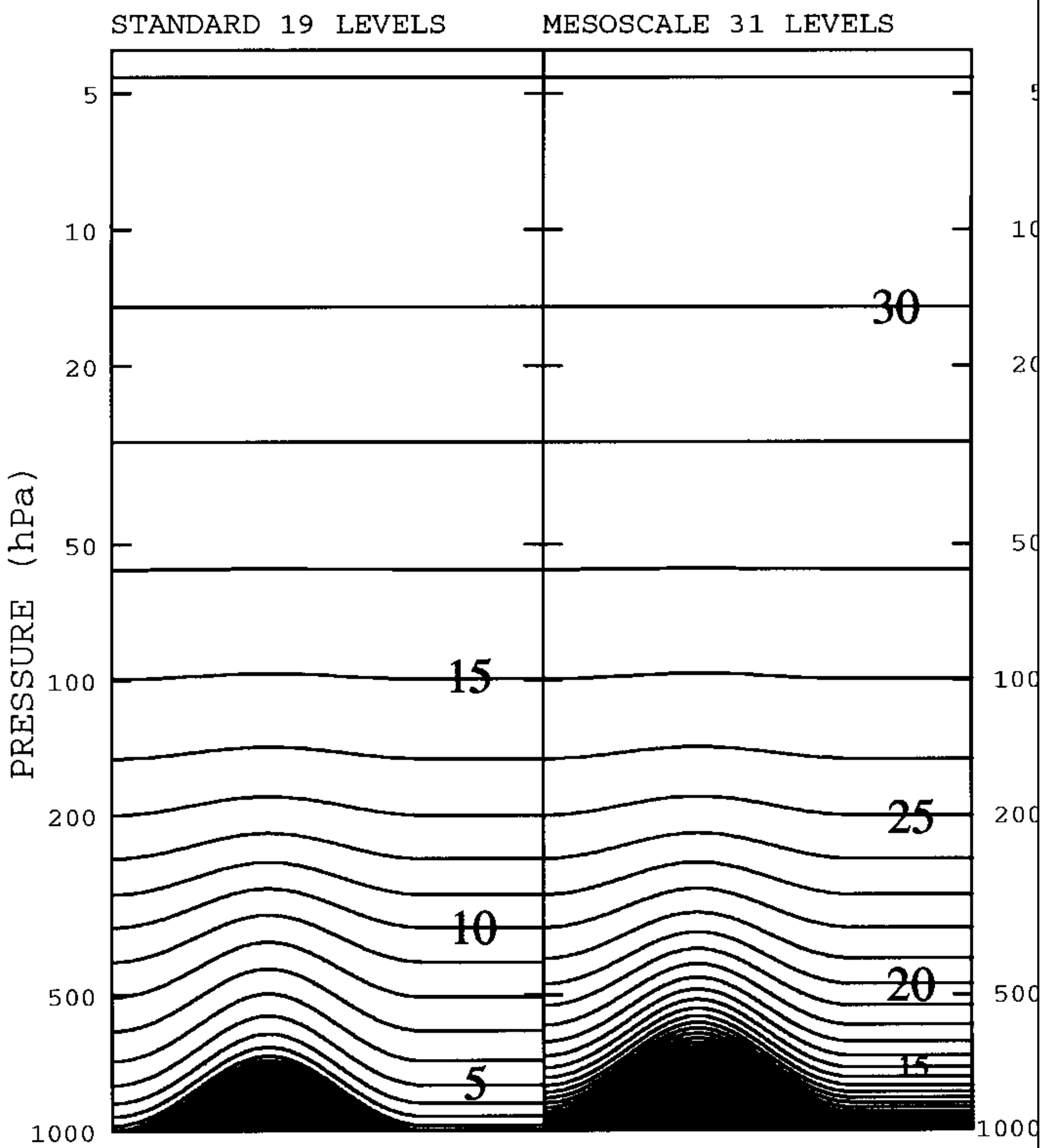
on a C grid. The point $P_{i,j+1/2}$ is the point where the v component of velocity would be held on a C grid.

Figure 1.b

B−p	C−u	B−p
C−v	B−u	C−v
B−p	C−u	B−p

Appendix 5 (incorporating appendix 4)

Appendix 6



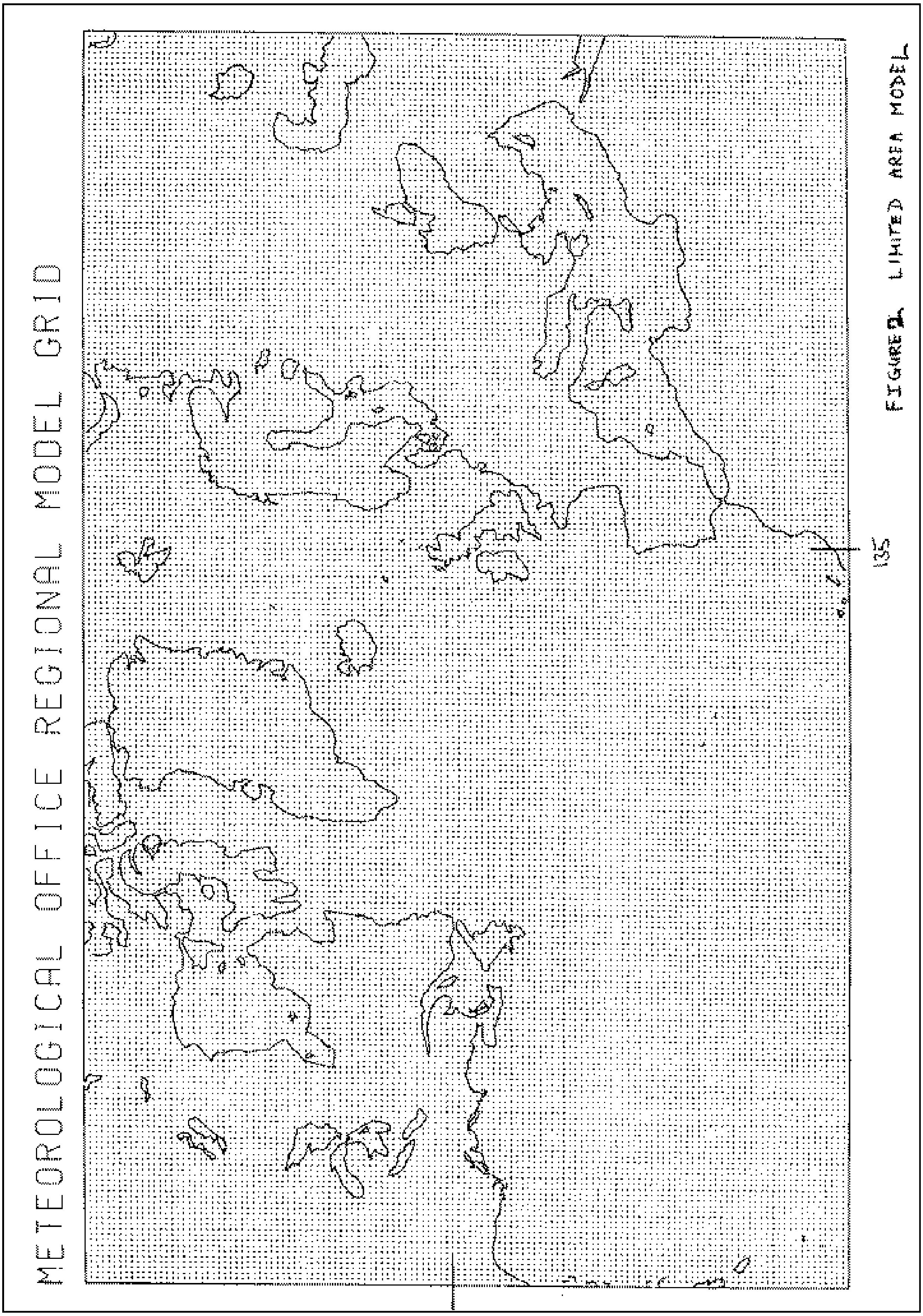


FIGURE 2. LIMITED AREA MODEL

METEOROLOGICAL OFFICE GLOBAL MODEL GRID

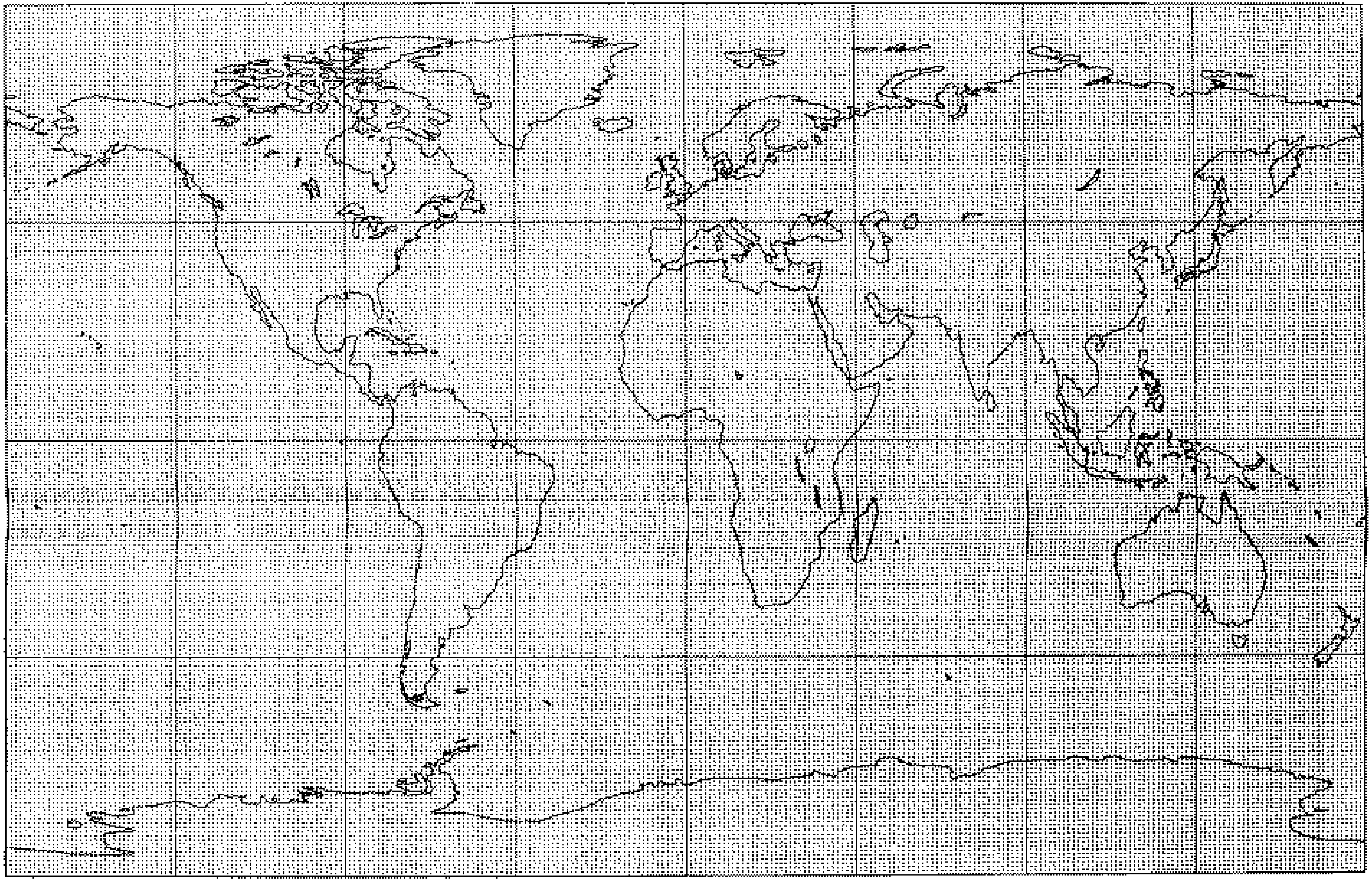


FIGURE 3. GLOBAL MODEL

FIGURE 4